Chapter I – Some fundamentals of semiconductors and optics

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Part I – Semiconductors

Why Semiconductors ?

- Semiconductors are the materials between metal and insulator.
- The conductivity can be flexibly adjusted by doping impurities.
- Different impurity gives different conductivity behaviors, such as hole and electron.
- Structure is almost perfect so that the materials are capable to be manipulated.
- The built-in field inside a semiconductor is controllable from outside.
- It's compact (small size) and has the capability for integration.

  - high-speed ICs, nano-technology applications, and quantum devices.
- Technology template (can be made by standard processing)

  - processing, packaging, and testing are ready and very suitable for mass production.

Resistivity, ρ (Ω cm) $\rho_{Insulator} > 10^4$ Ω cm $\rho_{Semiconductor} > 10^{-2} - 10^4$ Ω cm
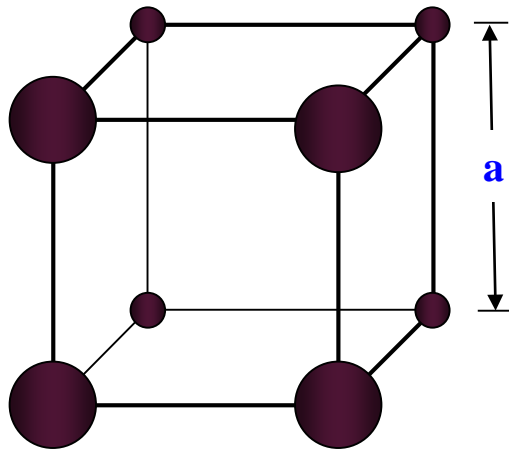
$\rho_{metal} < 10^{-6}$ Ω cm
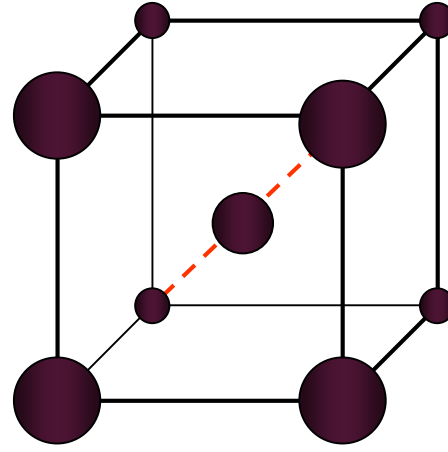
Periodic table of atoms
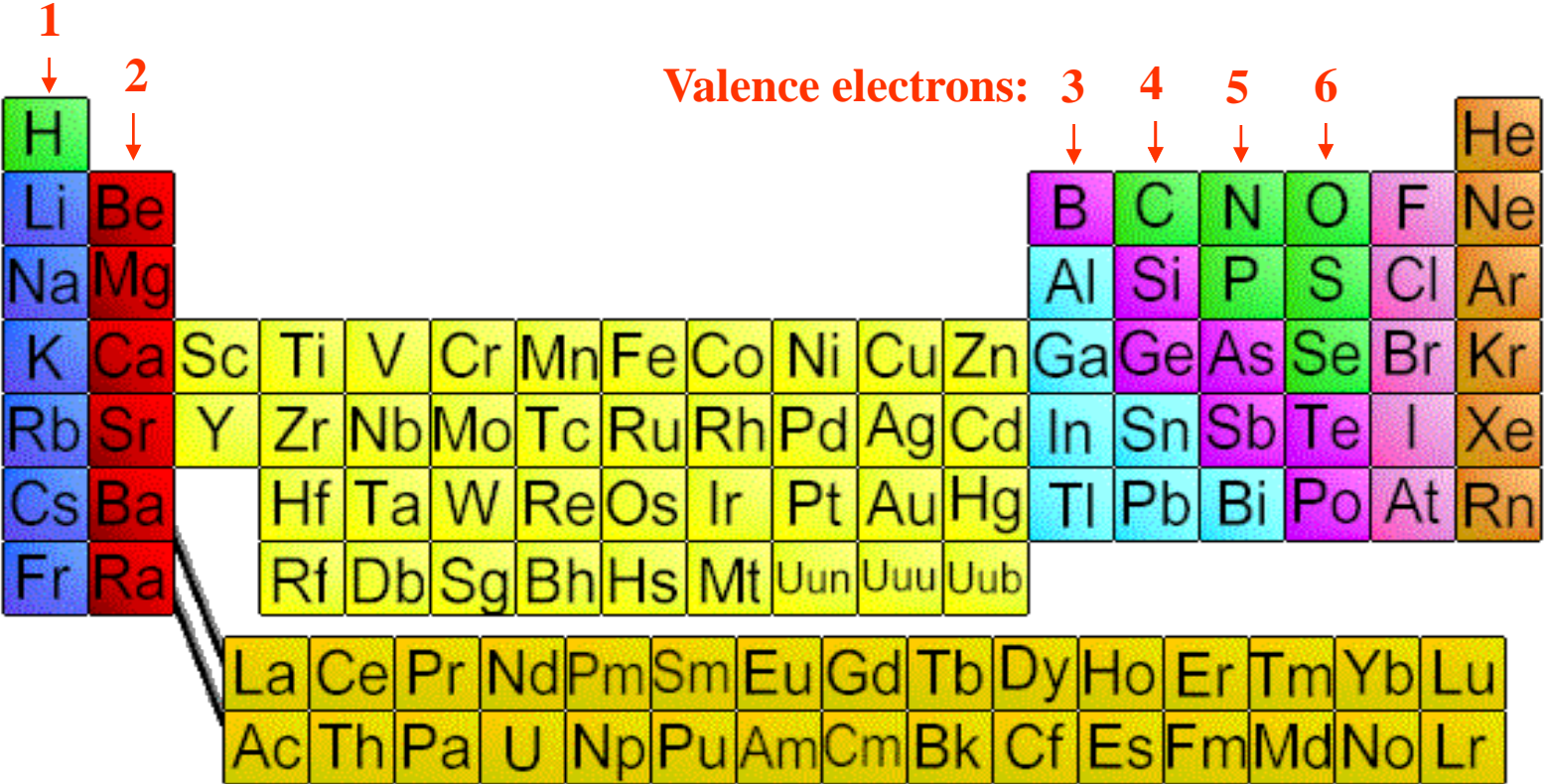
SiC, GaAs, BN, InAs, GaP, CdS, CdSe, ZnS, ZnSe, PbS, and PbTe
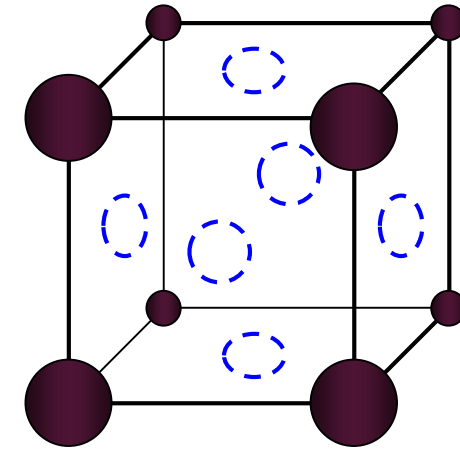
Crystal Structure

Cubic class of crystal

Simple cubic (s. c.)　　　　**Body center cubic (b. c. c.)**　　　　**Face center cubic (f. c. c.)**

- The semiconductor materials we will study are single crystal, that is, the atoms are arranged in a 3-D periodic fashion. The periodic arrangement of atoms in a crystal is called a lattice.

- For a given semiconductor, there is a unit cell that is representative of the entire lattice; by repeating the unit cell throughout the crystal, one can generate the entire lattice.

Miller index

Directions in crystals of the cubic class are very conveniently described in terms of Miller notation. For example, any plane in space can be represent by:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$ **where a, b, and c are the intercepts made by the plane at x, y, and z axes.**
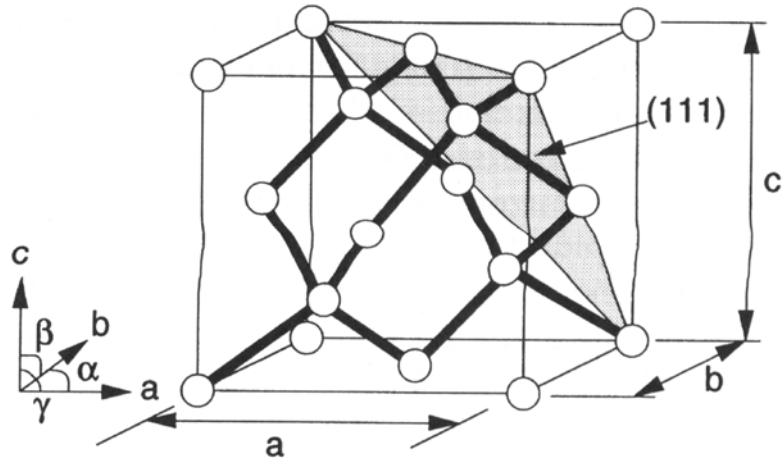
Writing h, k and l as the reciprocals of these intercepts, the plane can be described by
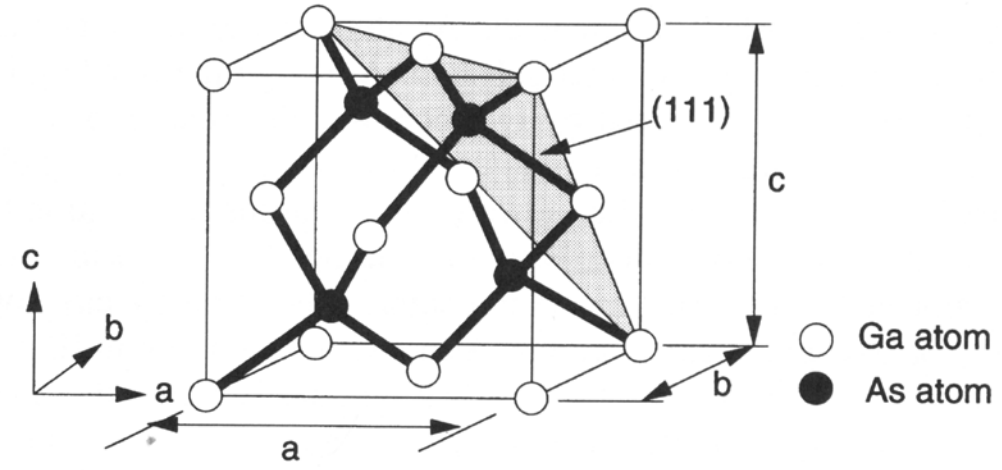
$$hx + ky + lz = 1$$

The Miller indices for this plane are written as (hkl).
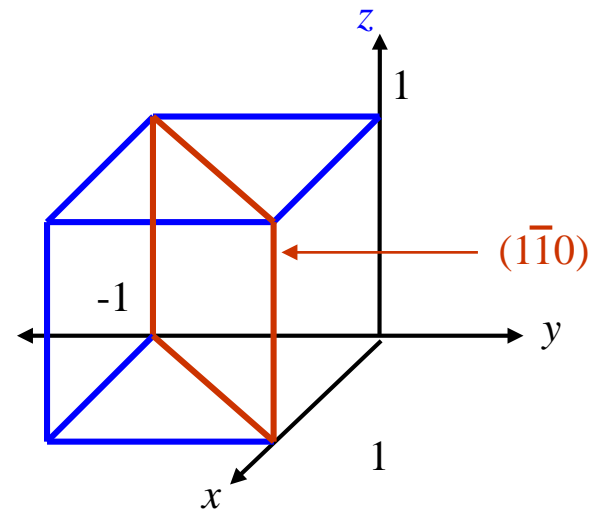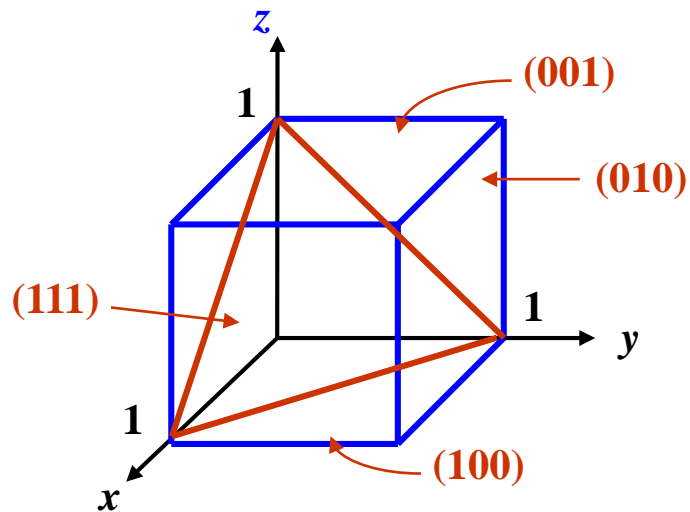
Atom arrangements of semiconductors

(a) Si (diamond) **Face center cubic**

○ Ga atom
● As atom

(b) GaAs (zincblende)

○ Ga atom
● N atom
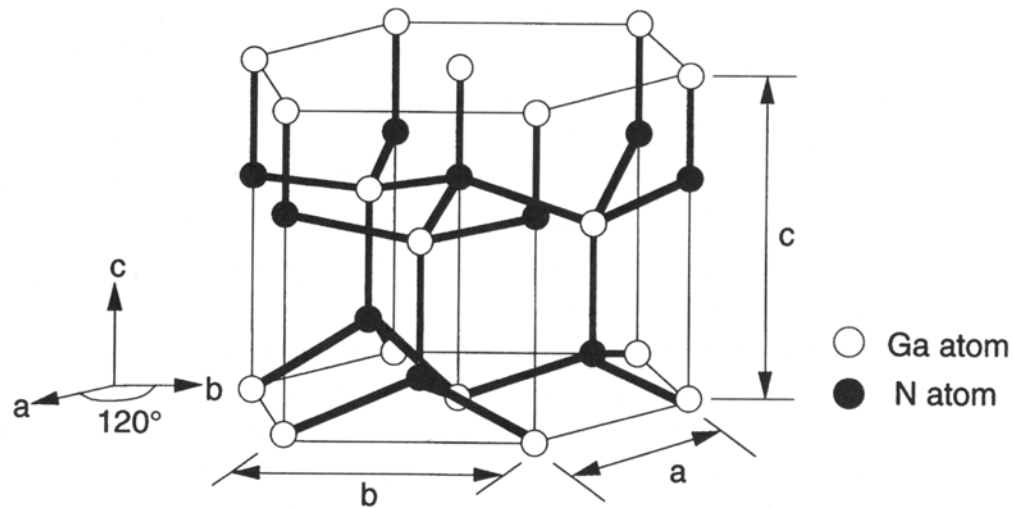
(c) GaN (wurtzite)

The ultimate tensile strength of Si (0.35×10^{10} dyn/cm^2) is maximum in <111> directions. In addition, the modulus of elasticity in the <111> directions is higher than that in the <110> or <100> directions (1.9×10^{12} dyn/cm^2, 1.7×10^{12} dyn/cm^2, and 1.3×10^{12} dyn/cm^2, respectively). As a result, Si tends to cleave on the {111} planes.

Energy Band of Semiconductors
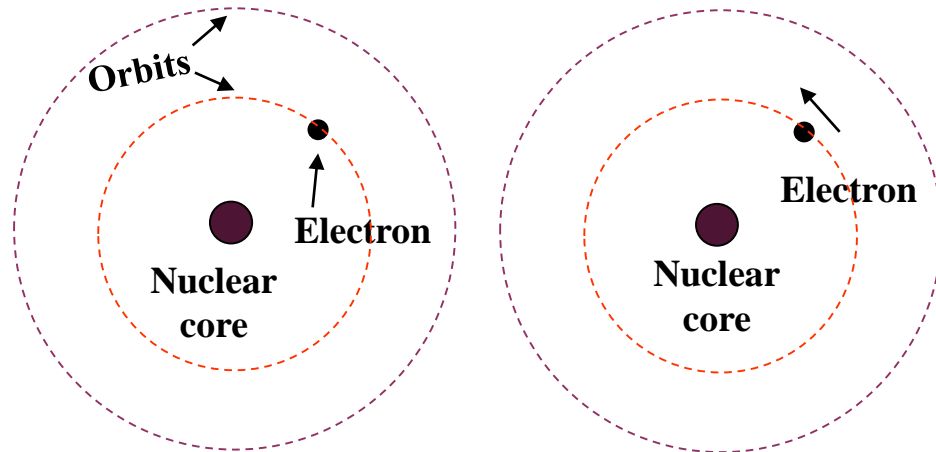
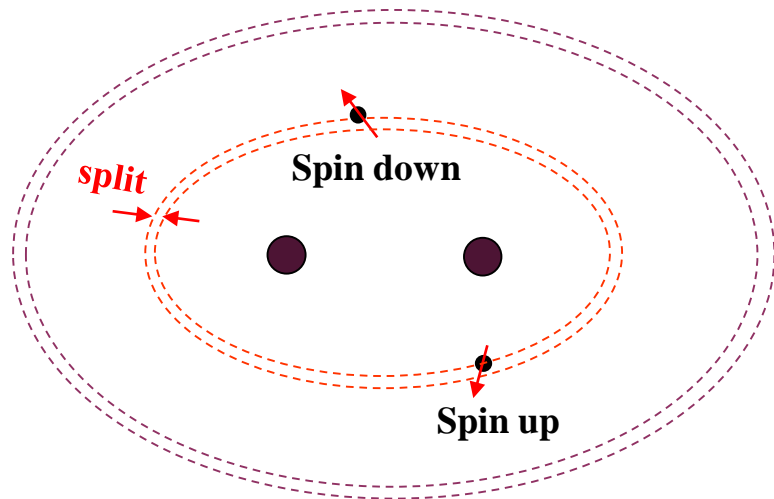

Two far apart atoms

Two close-up atoms

The energy level of an isolated hydrogen atom are given by Bohr model:

$$E_H = \frac{-m_0 q^4}{8\varepsilon_0^2 h^2 n^2} = \frac{-13.6}{n^2} eV$$

where n is a positive integer called the principle quantum number. The discrete energies are -13.6 eV for the ground level ($n = 1$), -3.4 eV for the first excited level ($n = 2$), etc.

The identical energy level of two closed-up atoms will split into two levels by the interaction between atoms. When we bring N atoms to form a crystal, the n-fold degenerate energy level will split into N separate but closely spaced levels due to atom interaction. This results in an essentially continuous band of energy.

Example: Alkali metals and insulators

Na

Na

Valence electron   Valence electron
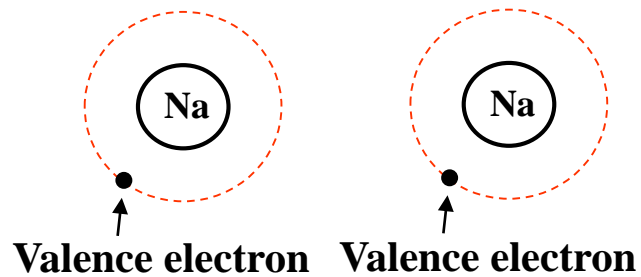
N Na atoms ➡ N valence electrons
But with 2N energy levels

Free electrons

2N energy levels

N valence electrons
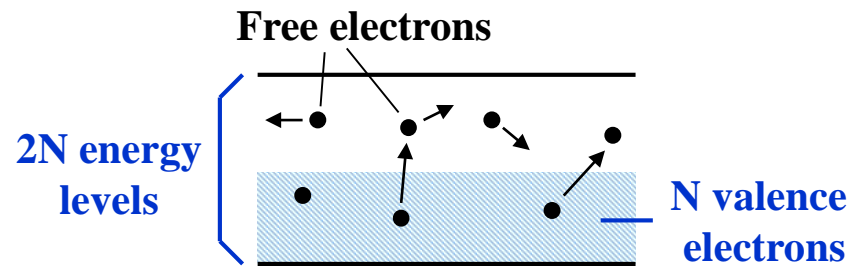
➡ High conductivity

For insulators, valence electrons completely fill an allowed energy band.

2N states empty

E_g

2N states full
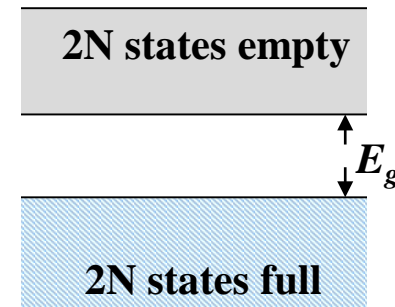
If E_g is too large, no electrons can gain enough energy to translate from the lower energy band to the higher energy band.

➡ Low conductivity

k-space diagram

For a free electron, the kinetic energy E is given by

$$E = \frac{p^2}{2m_0}$$ **where p is the particle momentum and m_0 is the free electron mass.**

However, the effective mass of a conduction band electron is different from the mass of a free electron. The energy-momentum relationship of a conduction electron can be written as

$$E = \frac{\bar{p}^2}{2m_n}$$ **where $\bar{p}$ is the crystal momentum and m_n is the free electron mass.**
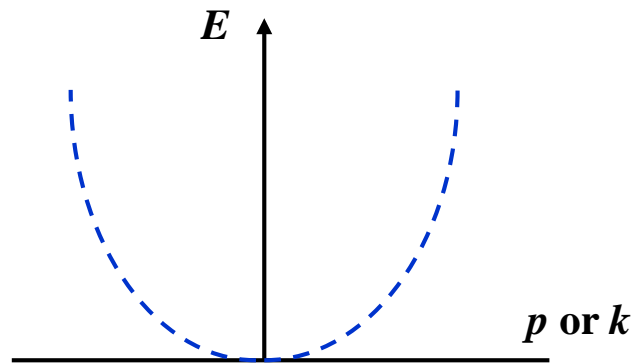
E-p (or k) plot of a free particle

f.f.c. crystal
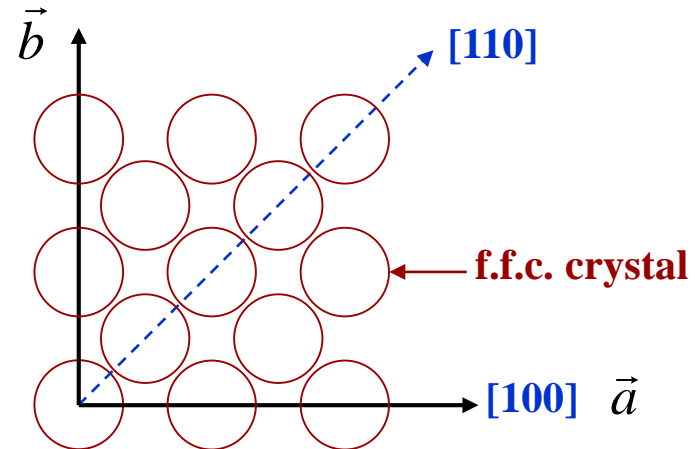
Electrons traveling in different directions encounter different potential patterns and therefore different k-space boundaries.

Energy band diagram
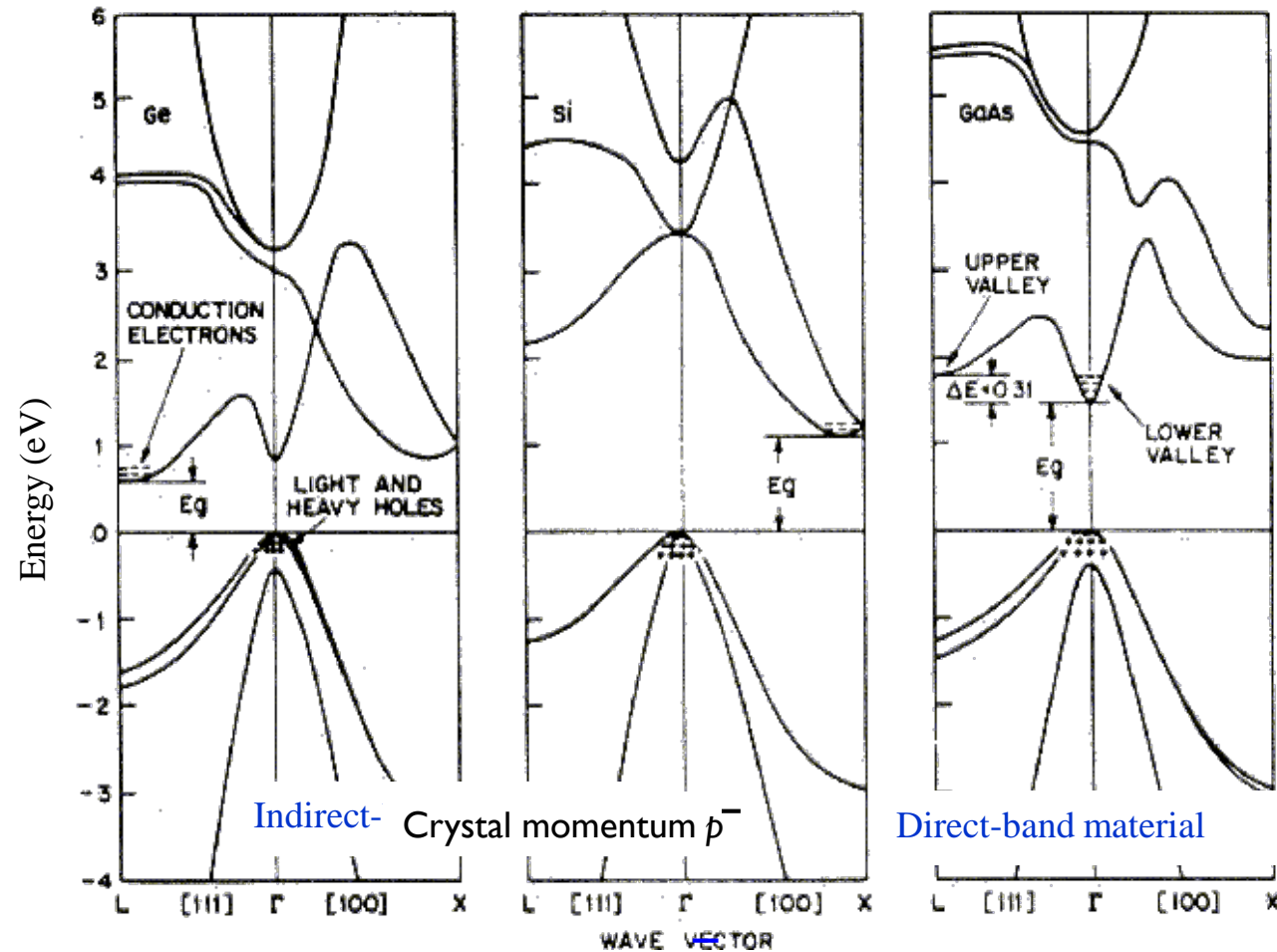
- **For direct band material, transitions between two allowed bands can take place with no change in crystal momentum p.**
- **A transition in an indirect bandgap material must necessarily include an interaction with the crystal so that crystal momentum is conserved.**

- **With a known *E-p* relationship, one cān obtain the effective mass from the second derivative of *E* with respect to *p***

$$m_n = \left[\frac{d^2 E}{dp^2}\right]^{-1}$$ ➡ **The narrower the parabola, the smaller the effective mass.**

Density of state of bulk semiconductors

$L/\lambda_x = m$ $L/\lambda_y = n$ $L/\lambda_z = k$ λ_i **: wavelength of electron, *i = x, y, z***

Use $\lambda_i = h/p_i$ ➡ $L{\times}p_x = h{\times}m$ or $L{\times}dp_x = h$
$L{\times}p_y = h{\times}n$ or $L{\times}dp_y = h$ ➡ $L^3{\times}dp_x{\times}dp_y{\times}dp_z = h^3$
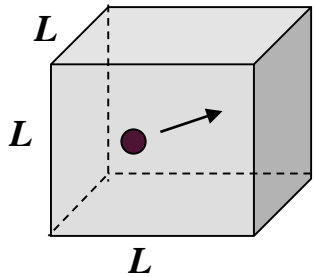$L{\times}p_z = h{\times}k$ or $L{\times}dp_z = h$

The volume of an allowed state in momentum space is h^3:

$$\therefore \quad N(E)dE = 2 \times \frac{4\pi p^2 dp}{h^3} = 4\pi \left(\frac{2m_n}{h^2}\right)^{3/2}\sqrt{E}dE \quad \Rightarrow \quad N(E) = 4\pi\left(\frac{2m_n}{h^2}\right)^{3/2}\sqrt{E}$$

Doping on Semiconductors
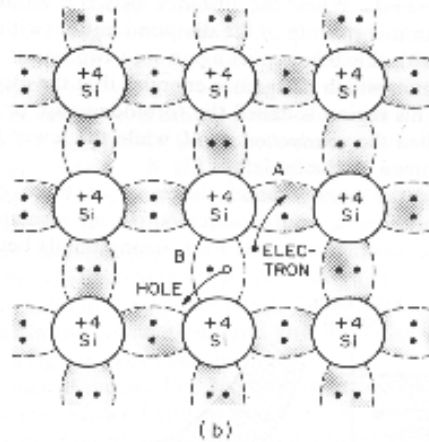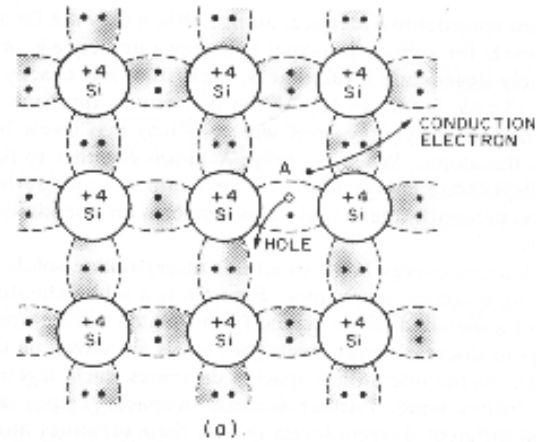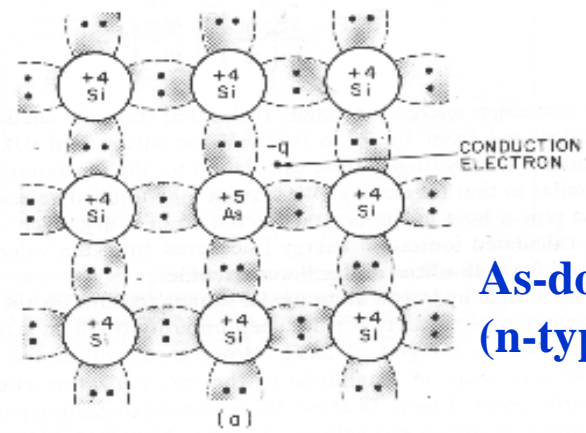
As-doped (n-type)
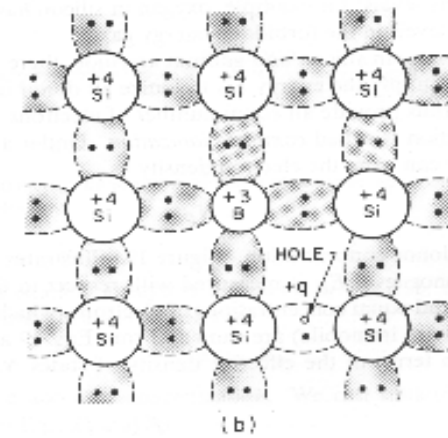
B-doped (p-type)

Pure Si-crystal   Doped Si-crystal

Activation of doping

Conduction band

E_d

E_v

**Donor doping
(n-type)**

Intrinsic

**Acceptor doping
(p-type)**

Valence band
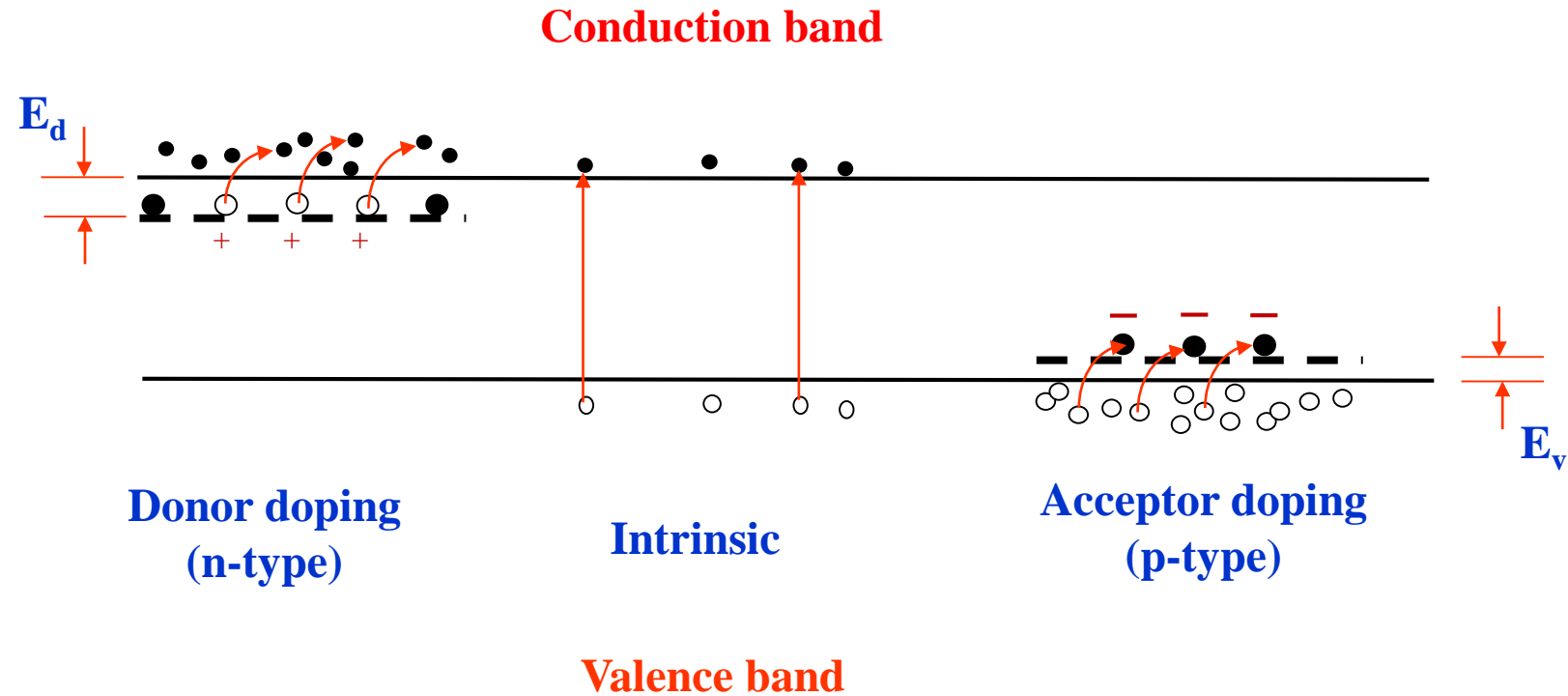
Due to the thermal agitating, the carrier concentration and the type of carrier can be controlled by impurity doping!!!

Ionization energies of impurities in Si

```
    |        |        |
 —  Si  ——  Si  ——  Si  —

    |        |        |

 —  Si  ——  •As  ——  Si  —

    |        |        |

 —  Si  ——  Si  ——  Si  —

    |        |        |
        n-type Si
```
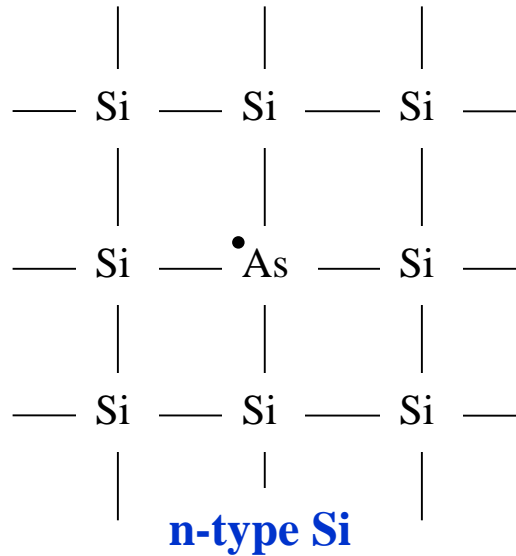
Energy levels of hydrogen atom are $E = -\dfrac{mq^4}{8a^2 h^2 \varepsilon_o^2}$

Using the energy required to remove an electron from the ground state of hydrogen atom is 13.6 eV,

$$E_{ionization} = \frac{13.6}{\varepsilon_r^2} \frac{m_n}{m_o}$$

for Si, $m_n = 0.06\ m_o$ and $\varepsilon_r = 11.8$, ➡ $E_{ionization} \sim 0.06$ eV

Ionization energies for group III and V impurities

Dopant	P	As	Sb	B	Al	Ga	In
Acceptor level				0.045	0.057	0.065	0.16
Donor level	0.044	0.049	0.039				

Fermi-Dirac distribution

Thermal equilibrium: the total energy of a semiconductor is a function only of crystal temperature.

Fermi-Dirac distribution : $f(E) = \dfrac{1}{1 + \exp(\dfrac{E - E_F}{kT})}$ E_F: **Fermi energy**

The probability of a state with energy E occupied by an electron at temperature T. The Fermi function only represents a probability of occupancy. It does not contain any information about the states available for occupancy and, therefore, can not by itself specify the electron population at a given energy.

$f(E)$
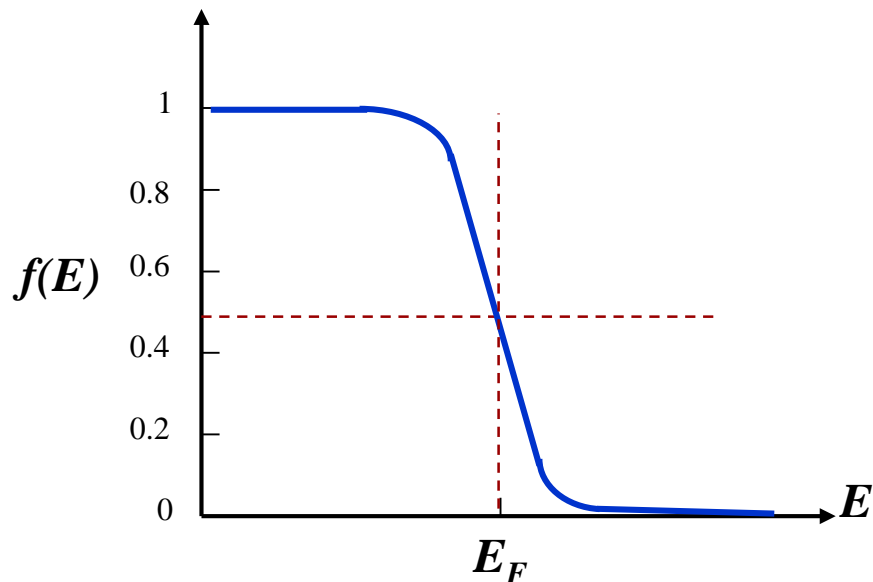
- E_F : **the highest filled state at T= 0°K (from the Pauli's exclusion principle)**

- **When $E = E_F$, $f(E) = 0.5$.**

- **Only the electrons near the Fermi's level have the contribution to current conduction.**

- $n = \int_0^{Etop} n(E)dE = \int_0^{Etop} N(E)f(E)dE$

Effective Density of States

The electron carrier density

$$n = \int N(E) \cdot f(E) \cdot dE \qquad \text{where} \qquad N(E) = 4\pi (\frac{2m_{e,h}}{h^2})^{3/2} \times \sqrt{E} \qquad \text{and} \qquad f(E) = \frac{1}{1 + \exp(\frac{E - E_F}{kT})}$$

Density of states   **Fermi-Dirac distribution**

If $E_{d,v} > 3kT$, $\rightarrow f(E) \approx e^{-(E - E_F)/kT}$ **or**   **If $E_{d,v} < 3kT$,** $\rightarrow f(E) \approx 1 - e^{-(E_F - E)/kT}$

$$n = \int N(E) \cdot f(E) \cdot dE \approx 4\pi \left[\frac{2m_e}{h^2}\right]^{3/2} \int_{E_C}^{\infty} \sqrt{E} \exp(-\frac{E - E_F}{kT}) dE$$

$$= 2\left[\frac{2\pi m_e kT}{h^2}\right]^{3/2} e^{-(E_C - E_F)/kT}$$

$$p = 2\left[\frac{2\pi m_p kT}{h^2}\right]^{3/2} e^{-(E_F - E_V)/kT}$$

$$n \approx N_C \times e^{-\frac{E_C - E_F}{kT}}$$

$$p \approx N_V \times e^{-\frac{E_F - E_V}{kT}}$$

where $N_{C,V} = 2 \times (\frac{2\pi m_{e,h} kT}{h^2})^{3/2}$

Effective density of state

If $E_{d,v} > 3kT, \rightarrow f(E) \approx e^{-(E-E_F)/kT}$

In n-type material that is not too highly doped, few of the allowed states in the conduction band are filled. The Fermi function in the conduction band is very small, and the Fermi level is well below the bottom of the conduction band. Then $(E_c - E_f) >> kT$, and the Fermi function reduces to the mathematically simpler Maxwell-Boltzmann distribution function. That is

$$f(E) \approx e^{-(E-E_F)/kT}$$

At energies well above the Fermi level the fraction of available states that are occupied is so small that the exclusive-principle limitation has no practical effect and Maxwell-Boltzmann statistics are applicable.

Free carriers of semiconductors

n-type

$E_c - E_F$

E_c

E_F

E_g

$E_c - E_F$

E_v

$E_c - E_F$ ↓   n ↗

$$n = N_C \times e^{-\frac{E_C - E_F}{kT}}$$

p-type

E_c

E_g E_F

E_v

$E_F - E_v$

$E_F - E_v$ ↓   p ↗
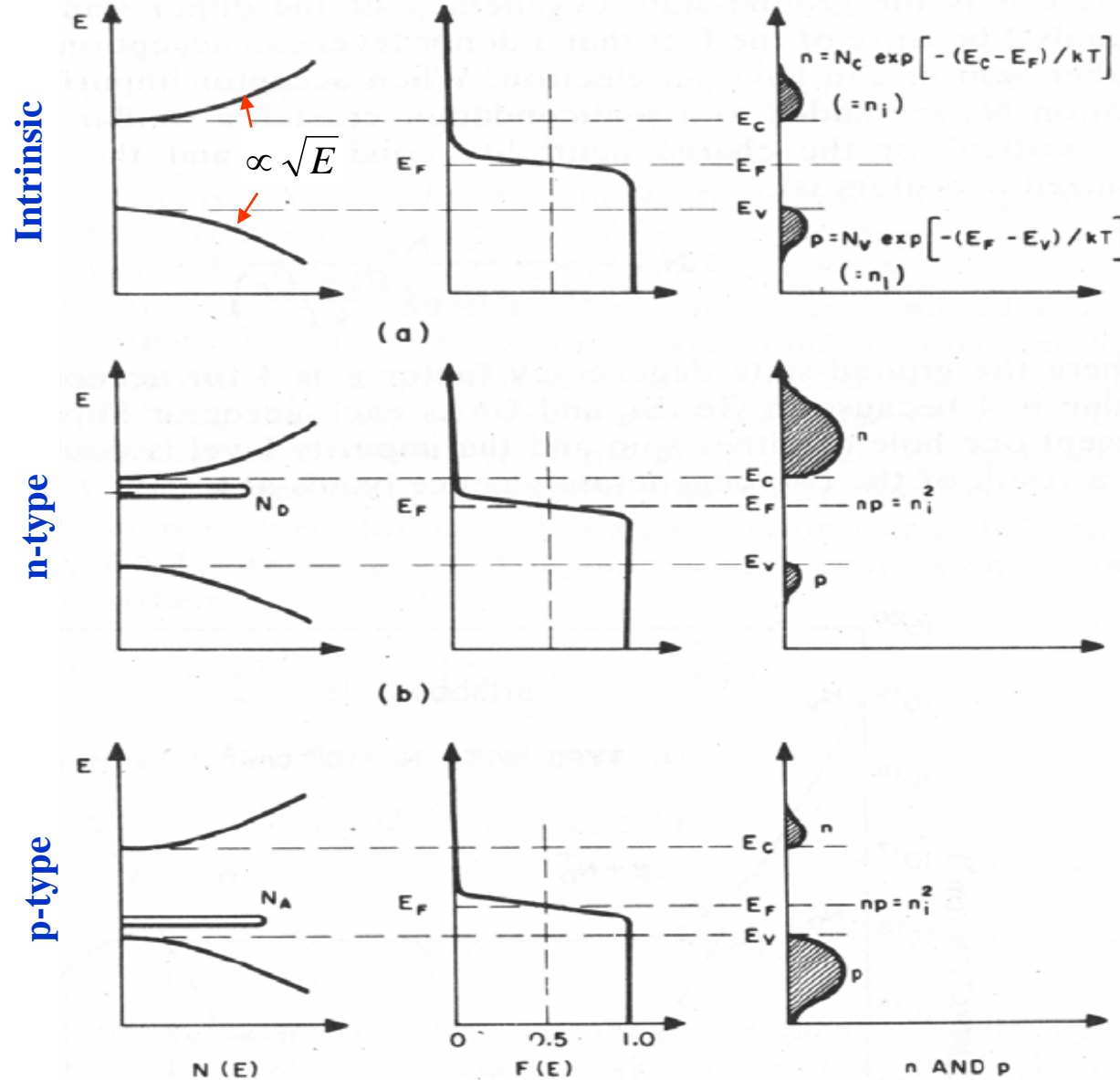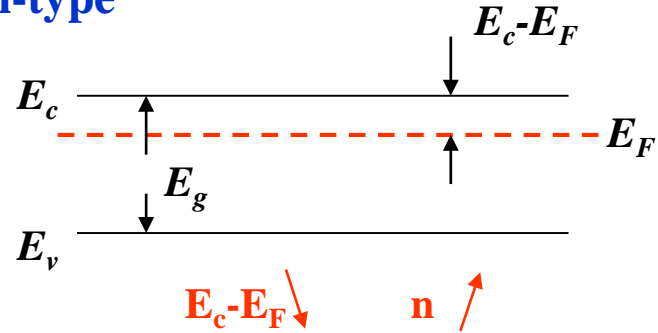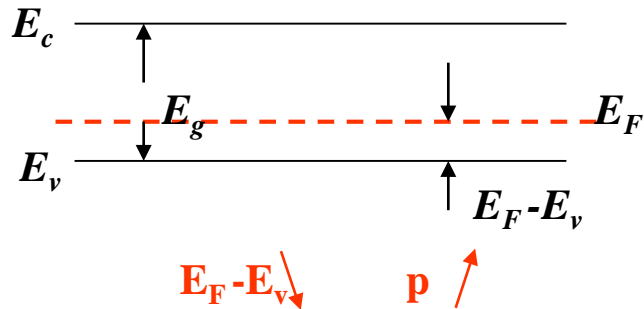
$$p = N_V \times e^{-\frac{E_F - E_V}{kT}}$$

States at energy E which are less than E_F are mostly filled; States at energy E which are larger than E_F are mostly unoccupied.

Example: Mass action law

$$n \approx N_C \times e^{-\frac{E_C - E_F}{kT}}$$

$$p \approx N_V \times e^{-\frac{E_F - E_V}{kT}}$$

$\Rightarrow$

$$n \cdot p = n_i^2 = N_c \cdot N_v \cdot \exp(-\frac{E_g}{kT})$$

At room temperature,

$kT \sim 25$ **meV** $\Rightarrow$ *exp(-3kT/kT) ~ 0.05*

$\Rightarrow$ n_i **(GaAs) = 1.2$\times$10^6 cm^{-3} < n_i (Si) = 1.45$\times$10^{10} cm^{-3}**

- **n, p are quite sensitive to the E_g , temperature and impurity level !!!**
- **The electrical properties can be controlled by impurity over a few orders of magnitude !!!**

Example: Intrinsic level

$$n = p \equiv n_i \to N_C \times e^{-\frac{E_C - E_F}{kT}} = N_V \times e^{-\frac{E_F - E_V}{kT}} \to E_F = \frac{E_C + E_V}{2} + \frac{kT}{2} \times \ln\left[\frac{N_V}{N_C}\right]$$

$$= \frac{E_C + E_V}{2} + \frac{3kT}{4} \times \ln\left[\frac{m_p}{m_e}\right]$$

At RT, the second term is much smaller than the E_g. Hence, the intrinsic Fermi level E_i lies very close to the middle of bandgap.

For n-type semiconductor

$$n_i = N_C \times e^{-\frac{E_C - E_i}{kT}}$$

$$N_D = N_C \times e^{-\frac{E_C - E_{Fn}}{kT}}$$

or

$$N_D = n_i \times e^{\frac{E_{Fn} - E_i}{kT}}$$

E_i: **intrinsic level**

E_{Fn}-E_i ↗ N_D ↗

For p-type semiconductor

$$p_i = N_V \times e^{-\frac{E_i - E_V}{kT}}$$

$$N_A = N_V \times e^{-\frac{E_{Fp} - E_V}{kT}}$$

or

$$N_A = p_i \times e^{\frac{E_i - E_{Fp}}{kT}}$$

E_i-E_{Fp} ↗ N_A ↗

Problem: **A Si ingot is doped with 10^{16} As atoms/cm³. Find the carrier concentration and Fermi level at 300K.**

At 300 K, we can assume complete ionization of impurity atoms. We have $n \sim N_D = 10^{16}$ cm⁻³

Use $p = \dfrac{n_i^2}{N_D} = 2.1 \times 10^4 \, cm^{-3}$

$$E_C - E_F = kT \times \ln\left[\dfrac{N_C}{N_D}\right] = 0.206eV \quad \textit{> 3kT}$$

$$E_F - E_i = kT \times \ln\left[\dfrac{N_D}{n_i}\right] = 0.354eV$$

If both donor and acceptor impurities present simultaneously, the impurity that is present in a greater concentration determines the type of conductivity in the semiconductor.

Example: Charge neutrality

If charge neutrality in a region where all dopant are ionized,

$$N_D + p = N_A + n \quad or \quad n - p = N_D - N_A$$

$$From \quad mass - action - law \rightarrow p = \frac{n_i^2}{n}$$

$$\rightarrow n - \frac{n_i^2}{n} = N_D - N_A \quad or$$

$$n = \frac{N_D - N_A}{2} + \sqrt{(\frac{N_D - N_A}{2})^2 + n_i^2} \quad \textbf{and} \quad p = \frac{N_A - N_D}{2} + \sqrt{(\frac{N_A - N_D}{2})^2 + n_i^2}$$

In general, the net impurity concentration $|N_D - N_A|$ is larger than the intrinsic carrier concentration. Therefore the above equations can be simplified to

$$n \approx N_D - N_A \quad or \quad p \approx N_A - N_D$$

Problem: Compensation

A silicon crystal is known to contain 10^{-4} atomic percentage of arsenic (As) as an impurity. It then receives a uniformity doping of 3×10^{16} cm^{-3} phosphorus (P) atoms and subsequent uniformity doping of 10^{18} cm^{-3} boron (B) atoms. All dopant are activated by thermal annealing.

Si has 5×10^{22} atoms cm^{-3}, 10^{-4} atomic percentage

➡ Si is doped to a concentration of $5 \times 10^{22} \times 10^{-6} = 5 \times 10^{16}$ As atoms cm^{-3}

The added doping of 3×10^{16} P atoms cm^{-3},

➡ $5 \times 10^{16} + 3 \times 10^{16} = 8 \times 10^{16}$ n-type atoms cm^{-3}

p-type doping is 10^{18} cm^{-3},

➡ $p = 10^{18} - 8 \times 10^{16} = 9.2 \times 10^{17}$ cm^{-3}

Carrier transport phenomena

Carrier Drift

Mobility

Under thermal equilibrium, the average thermal energy of a conduction electron can be given by:

$$\frac{1}{2}m_e v_{th}^2 = \frac{3}{2}kT$$ **where v_{th} is the average thermal velocity.**

At 300 K, the thermal velocity is about 10^7 cm/s for Si and GaAs. Electrons in semiconductor are therefore moving rapidly in all direction. The thermal motion of an electron can be visualized as a succession of random scattering from collisions with lattices, impurities, and other scattering centers. The average distance between collisions is called mean free path, and the average time between collision is called mean free time, τ_{cn}.

$$\vec{F} = 0 \qquad \vec{F} \neq 0$$

τ_{cn}

τ_{cn}: scattering mean-free time
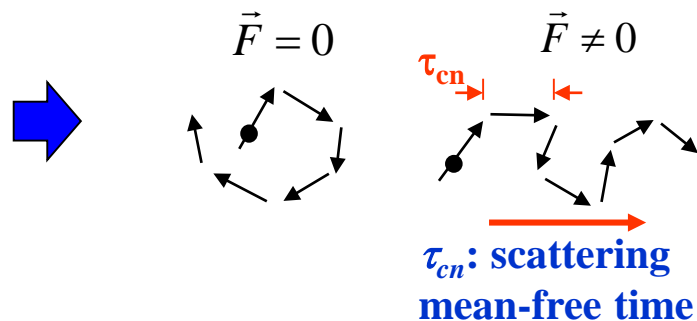
Electrons exchange energy when collide with the lattice and drop toward their thermal equilibrium positions. The energy lost in each collision is much less than the mean thermal energy of the electrons.

$$F \cdot \Delta t = m \cdot \Delta v$$

$$-qE\tau_{cn} = m_n^* \cdot v_d \qquad \text{or} \quad v_d = \frac{-qE\tau_{cn}}{m_n^*} \qquad \text{where } v_d \text{ is the drift velocity in an applied field.}$$

Let mobility $\mu_n = -\dfrac{v_d}{E} = q \times \dfrac{\tau_{cn}}{m_n^*}$

Mobility is an important parameter for carrier transport because it describes how strongly the motion of an electron is influenced by an electric field.

Example: hot carriers

For Si semiconductor, electrons with μ_n = 1400 cm²/Vsec have a drift velocity of only 1.5 % of the thermal velocity if E = 100 V/cm. Typical thermal velocity at RT is 10^7 cm/sec.

$$\mathbf{E} \nearrow \quad v_d \nearrow \quad \mathbf{E_k} = \frac{1}{2} m_n^* v_d^2 \nearrow$$

When electrons attain energy above the ambient thermal energy, they are often characterized as hot carriers with an effective temperature T_e. When the energy of the hot carriers reaches a critical value, a new scattering process (optical phonon scattering) become important. This new scattering process can effectively transfer energy from the hot carriers to the lattice, and it is the major reason that the drift velocity approaches a limit value at high field conditions.

Current density:

$$\vec{J}_n = \sum_i q\vec{v}_i = -nq\vec{v}_d = nq\mu_n\vec{E}$$

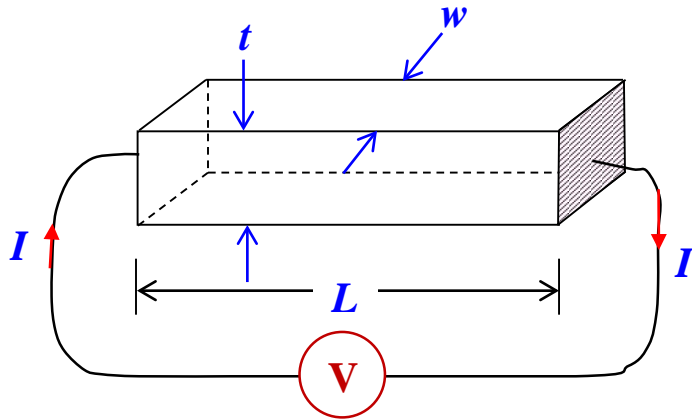$$\vec{J}_p = pq\mu_p\vec{E}$$

$$\vec{J}_{tot} = \vec{J}_n + \vec{J}_p$$

Empirical expression of *e-h* mobility

$$\mu_n = 88 \times T_n^{-0.57} + \frac{7.4 \times 10^8 T^{-2.33}}{1 + [N/(1.26 \times 10^{17} T_n^{2.4})] \times 0.88 \times T_n^{-0.146}}$$

$$\mu_p = 54.3 \times T_n^{-0.57} + \frac{7.4 \times 10^8 T^{-2.23}}{1 + [N/(2.35 \times 10^{17} T_n^{2.4})] \times 0.88 \times T_n^{-0.146}}$$

N (cm^{-3})	Arsenic	Phosphorus	Boron
10^{13}	1423	1424	486
10^{14}	1413	1416	485
10^{15}	1367	1374	478
10^{16}	1184	1194	444
10^{17}	731	727	328
10^{18}	285	279	157
10^{19}	108	115	72

Resistivity

$$I = JA = qnv_d A = qn\mu_n EA = qn\mu_n \frac{V}{L} A = \sigma \frac{A}{L} V$$

$$J = J_n + J_p = (qn\mu_n + qp\mu_p) \cdot E$$

$$\rho = \frac{1}{\sigma} = \frac{1}{qn\mu_n + qp\mu_p}$$

$$n, p \uparrow \quad \mu_{n,p} \uparrow \quad \Rightarrow \quad \rho \downarrow$$

Carrier concentration and mobility are the two most important factors which affect the conductivity of a semiconductor.

$$R = \rho \frac{L}{A} = \rho \frac{L}{t \times w}$$

Let's choose $L = w \rightarrow$

$$R = \rho \frac{L}{A} = \rho \frac{w}{t \times w} = \rho \frac{1}{t} \equiv R_s \quad \Omega/\square$$

Sheet resistance

Four-point probe

The four-point probe technique can also be used to measure resistance of a connection where contact resistance (R_c) of probes can not be ignored. As shown in the following figure, probe 1 and probe 2 are used to inject current to the contact pads, and probe 3 and probe 4 are used to measure the voltage difference between the two pads. Please show that the resistance (R_l) between the two pads can be readily obtained by dividing the voltage measured by probe 3 and 4 to the current injected by probe 1 and 2. The contact resistance of the probes are canceled by using the technique.
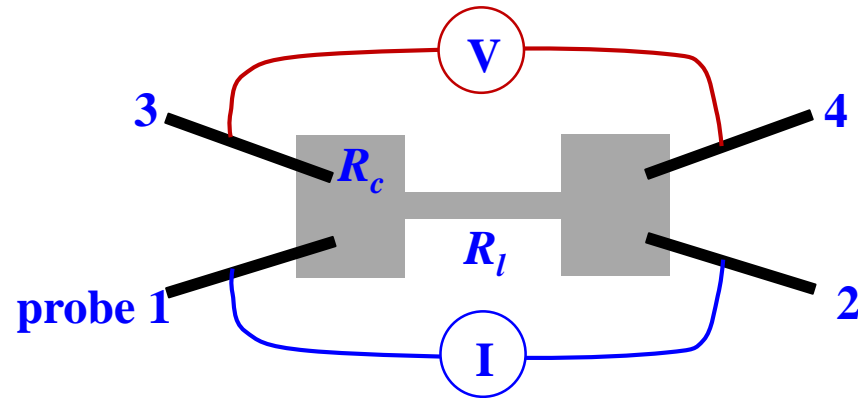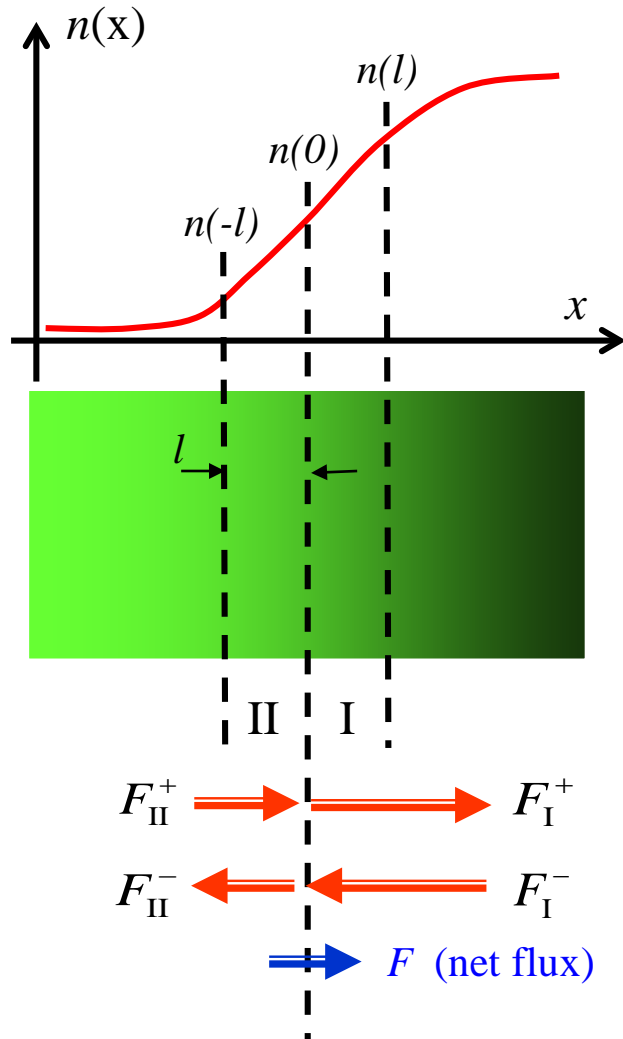

Top view
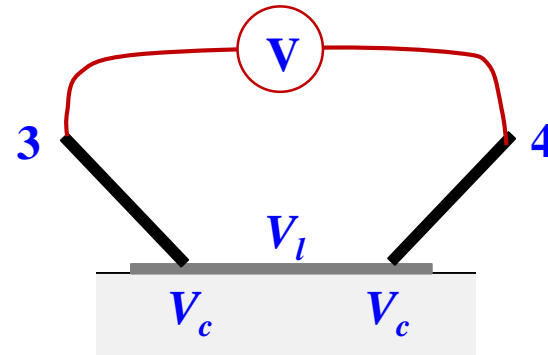
Side view

Carrier Diffusion

Because of the finite temperature, the electrons have random motions. The gradient of carrier concentration will result in a net carrier flow. Diffusion current is generally not an important consideration in metals because of their high conductivity. The lower conductivity and the possibility of non-uniform density of carriers and of carrier energies, however, often makes diffusion a very important process affecting current flow in semiconductors.
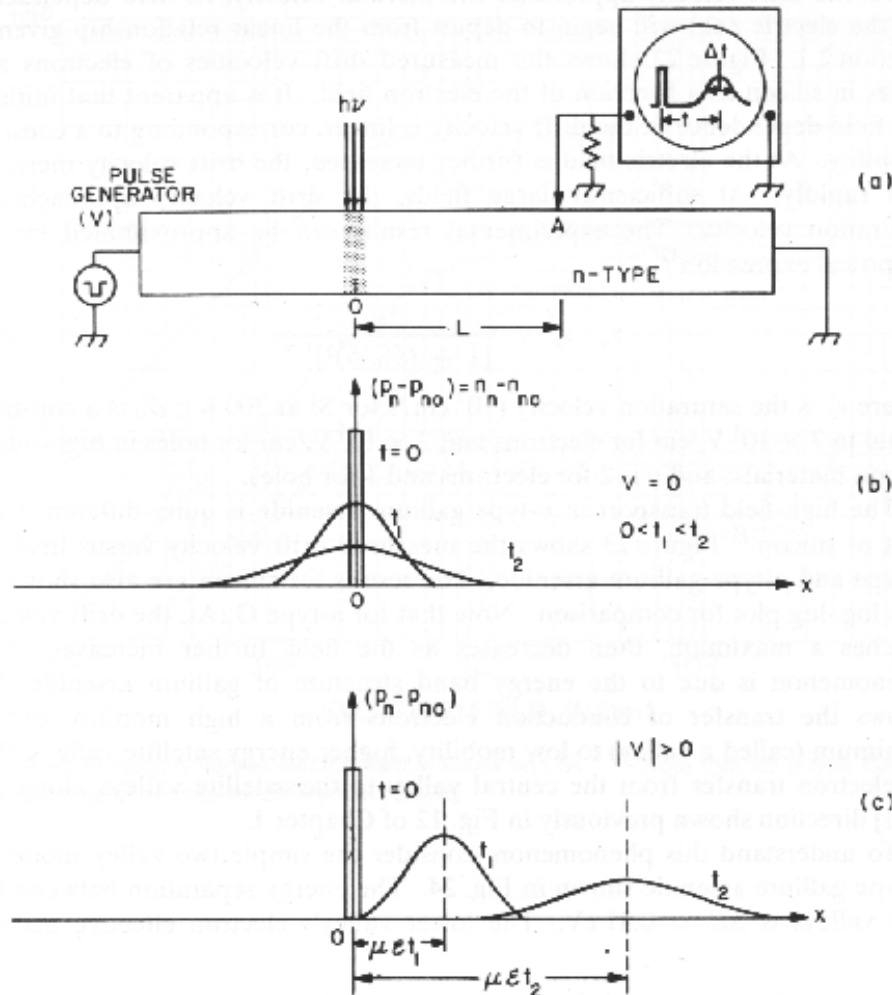
$$F = \frac{1}{2}v_{th}[n(-l) - n(l)]$$

$$F = \frac{1}{2}v_{th}([n(0) - l\frac{dn}{dx}] - [n(0) + l\frac{dn}{dx}])$$

$$= -v_{th}l\frac{dn}{dx} = -D_n\frac{dn}{dx} \quad D_n : \text{Diffusion constant}$$

$$J_n = -qF = qD_n\frac{dn}{dx} \qquad J_p = -qD_p\frac{dp}{dx}$$

Current density equations

The carrier-transport process involves not only the drift and also the diffusion processes !!!
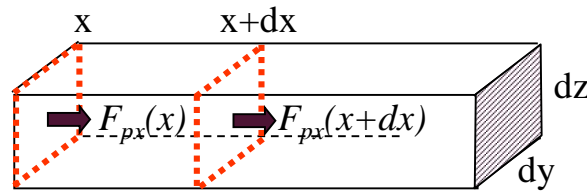
Total current is:

$$J_n = nq\mu_n E + qD_n \frac{dn}{dx} \qquad J_p = pq\mu_p E - qD_p \frac{dp}{dx}$$

$$J_{cond} = J_n + J_p$$

Continuity Equations

We are now ready to consider the over all effect when drift, diffusion, and recombination occur simultaneously in a semiconductor material.


$$F_{px}(x+dx) = F_{px}(x) + dF_{px}/dx \bullet dx$$

$$\frac{\partial p}{\partial t}dxdydz = \{F_{px}(x) - F_{px}(x+dx)\}dydz = -\frac{\partial F_{px}}{\partial x}dxdydz$$

The number of hole in the slice may increase due to net current flow into the slice and net carrier generation in the slice. The over all rate of holes increase include three components: the change of hole flux at x, plus the rate at which holes are generated, minus the rate at which they are recombined with electrons in the slice.

$$\frac{\partial p_n}{\partial t}dxdydz = -\frac{\partial F_{px}}{\partial x}dxdydz + G_p dxdydz - \frac{p_n - p_{n0}}{\tau_p}dxdydz$$

Generation: G_p Recombination: R_p
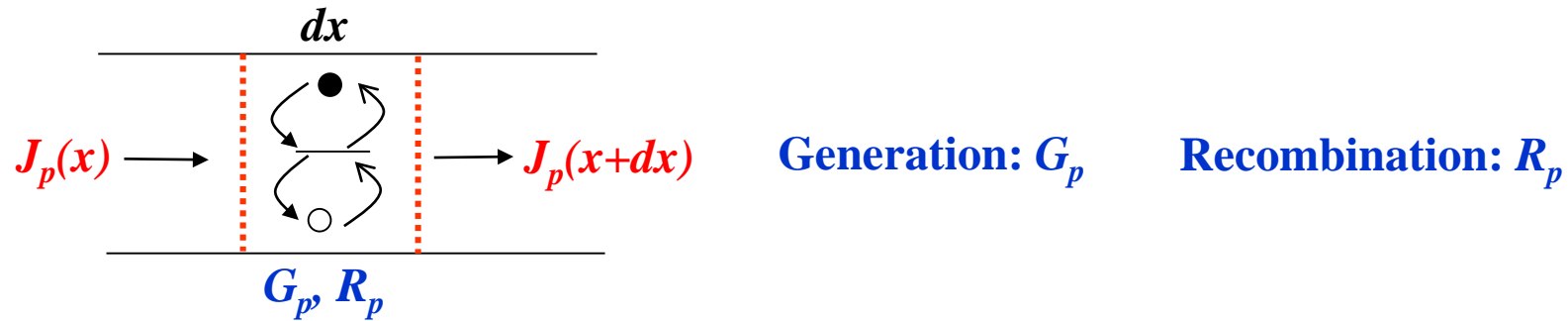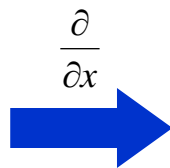
$$\frac{\partial p_n}{\partial t}dxdydz = -\frac{\partial F_{px}}{\partial x}dxdydz + G_p dxdydz - \frac{p_n - p_{n0}}{\tau_p}dxdydz$$

$$\frac{\partial p_n}{\partial t} = -\frac{\partial F_{px}}{\partial x} + G_p - \frac{p_n - p_{n0}}{\tau_p} \quad \textbf{and} \quad \frac{\partial n_p}{\partial t} = -\frac{\partial F_{nx}}{\partial x} + G_n - \frac{n_p - n_{p0}}{\tau_n}$$

$$\frac{J_n}{q} = n\mu_n E + D_n\frac{dn}{dx}$$

$$\frac{J_p}{q} = p\mu_p E - D_p\frac{dp}{dx}$$

$\frac{\partial}{\partial x}$

$$\frac{1}{q}\frac{\partial J_n}{\partial x} = \mu_n\frac{\partial}{\partial x}(nE) + D_n\frac{\partial^2 n}{\partial x^2} = -\frac{\partial F_{nx}}{\partial x}$$

$$\frac{1}{q}\frac{\partial J_p}{\partial x} = -\mu_p\frac{\partial}{\partial x}(pE) + D_p\frac{\partial^2 p}{\partial x^2} = -\frac{\partial F_{px}}{\partial x}$$
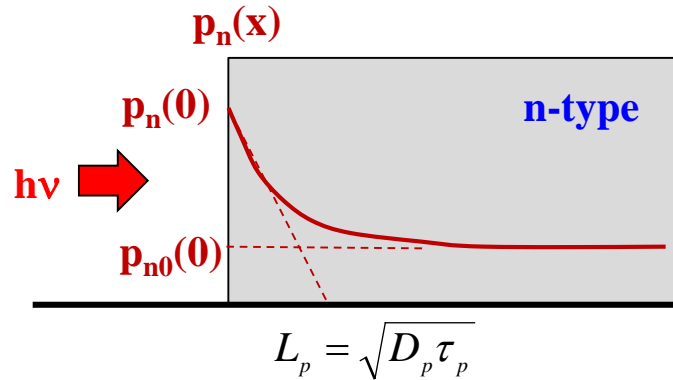
$$\frac{\partial n_p}{\partial t} = n_p \mu_n \frac{\partial E}{\partial x} + \mu_n E \frac{\partial n_p}{\partial x} + D_n \frac{\partial^2 n_p}{\partial x^2} + G_n - \frac{n_p - n_{po}}{\tau_n}$$

$$\frac{\partial p_n}{\partial t} = -p_n \mu_p \frac{\partial E}{\partial x} - \mu_p E \frac{\partial n_p}{\partial x} + D_p \frac{\partial^2 p_n}{\partial x^2} + G_p - \frac{p_n - p_{no}}{\tau_p}$$

In addition to the continuity equations, Poisson equation must also be satisfied.

$$\frac{\partial E}{\partial x} = \frac{\rho_s}{\varepsilon_s} \qquad \textbf{and} \qquad \rho_s = q \times (p - n + N_D^+ - N_A^-)$$

where ρ_s is the space charge density and ε_s is the dielectric permittivity.
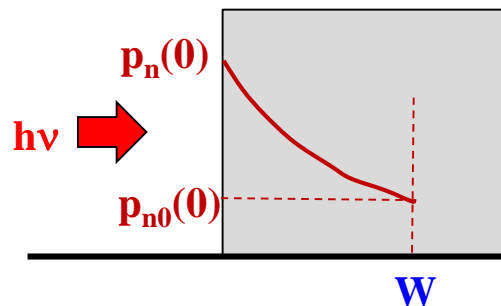
Steady-state injection from one side

At steady state,

$$\frac{\partial p_n}{\partial t} = 0 = D_p \frac{\partial^2 p_n}{\partial x^2} - \frac{p_n - p_{n0}}{\tau_p}$$

and the B.C. are: $p_n(x = 0) = p_n(0)$ and $p_n(\infty) = p_{n0}$

The solution of $p_n(x)$ is $p_n(x) = p_{n0} + [p_n(0) - p_{n0}]e^{-x/L_p}$ **where L_p is called the diffusion length.**

If the boundary conditions are changed so that all excess carriers at $x = W$ are extracted, that is, $p_n(W) = p_{n0}$, then the solution becomes

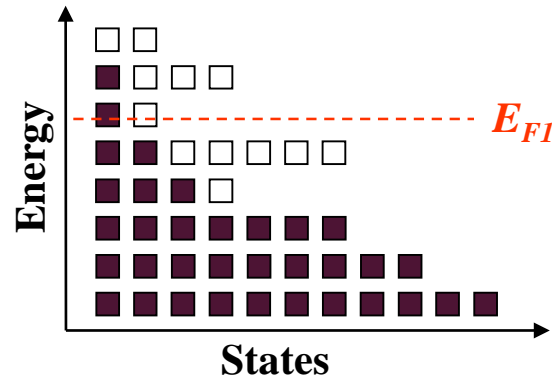$$p_n(x) = p_{n0} + [p_n(0) - p_{n0}]\frac{\sinh[W - x/L_p]}{\sinh(W/L_p)}$$

$$J_p = -qD_p \frac{\partial p_n}{\partial x}\Big|_W = q[p_n(0) - p_{n0}]\frac{D_p}{L_p}\frac{1}{\sinh(W/L_p)}$$
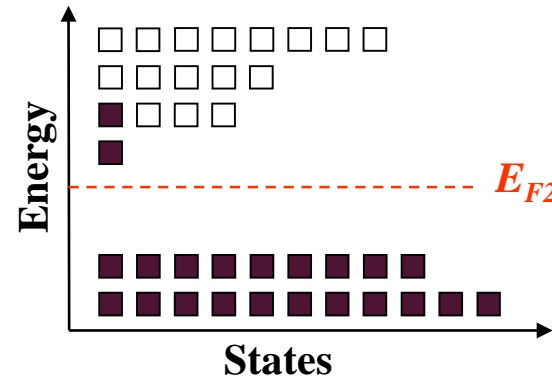
Metal-semiconductor Contacts

Equilibrium in Electronic Systems

At thermal equilibrium, Fermi level is constant throughout a system.

System 1 (metal)

System 2 (semiconductor)

The filled state density is

$$n = g \bullet F_D$$

Fermi function

The vacant state density is

$$v = g \bullet (1 \text{-} F_D)$$
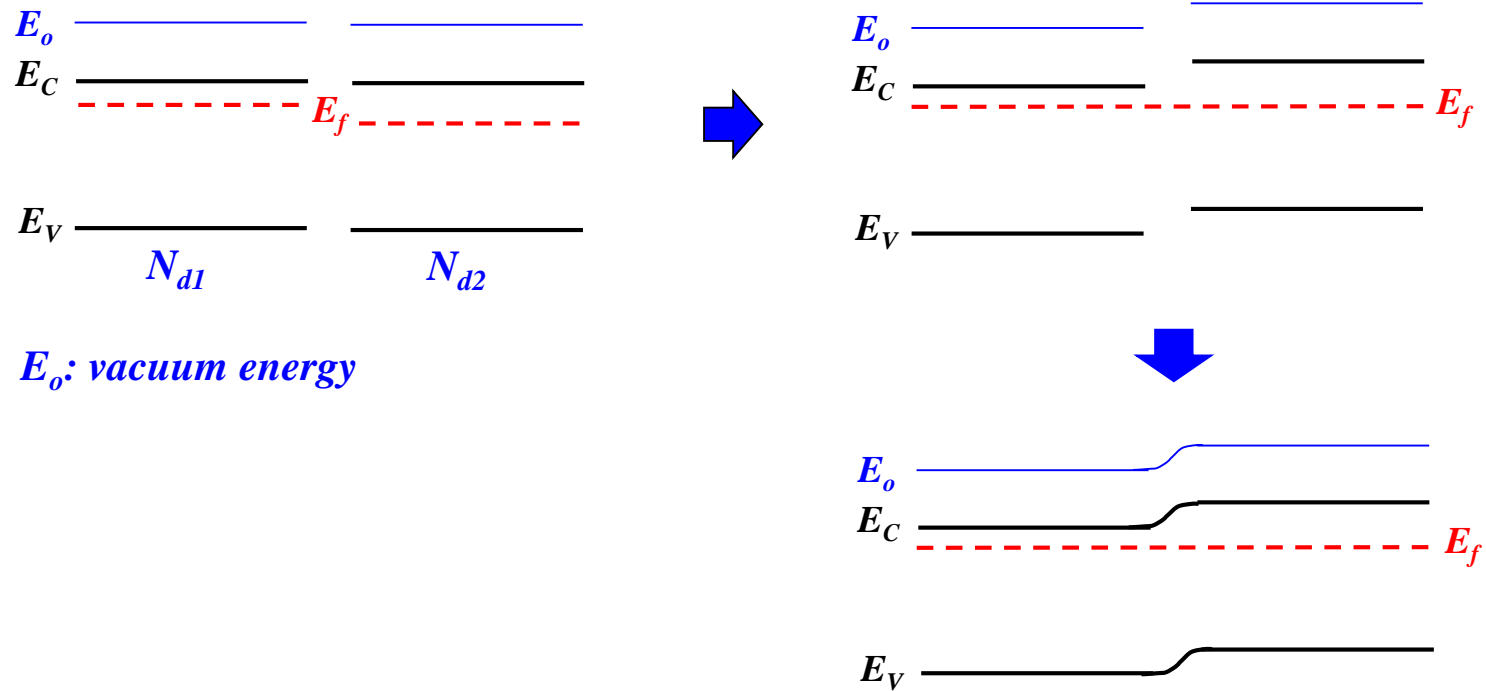
Allowed electron state density

We now consider that the two systems are brought in intimate contact. Equilibrium is reached when there is no net transfer of electrons at any energy.

$$\therefore \quad n_1 v_2 = n_2 v_1 \quad or \quad F_{D1} g_1 (1 - F_{D2}) g_2 = F_{D2} g_2 (1 - F_{D1}) g_1$$

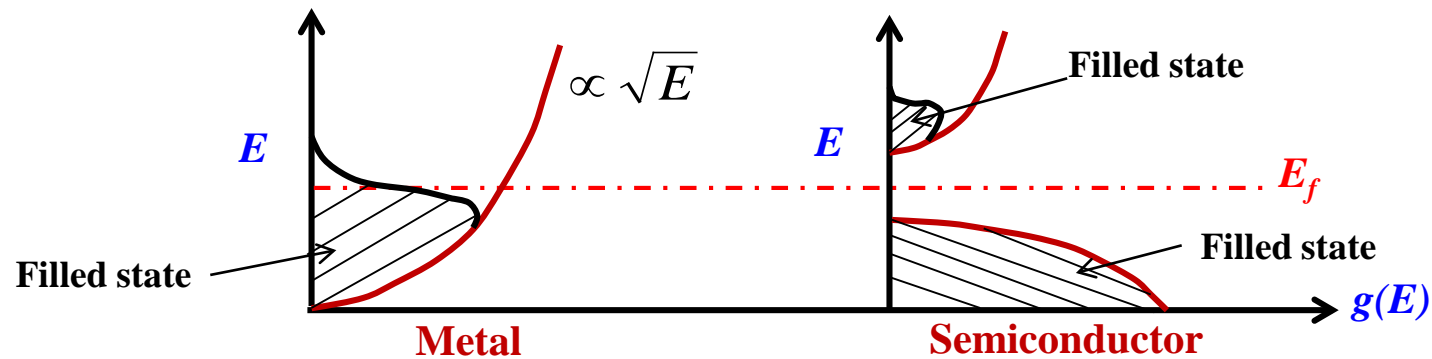$$\Rightarrow \quad F_{D1} = F_{D2} \quad or \quad E_{F1} = E_{F2}$$

Example: inhomogeneously doped semiconductor

Consider now the energy band diagram at thermal equilibrium for an n-type semiconductor doped with N_{d1} for $0 < x < a$ and with N_{d2} for $x > a$. Please draw the energy band diagram of this system.


E_o: vacuum energy

Band diagram

In the metal, the Fermi level is immersed within a continuum of allowed states, while in semiconductor, the density of state is negligible at the Fermi level.
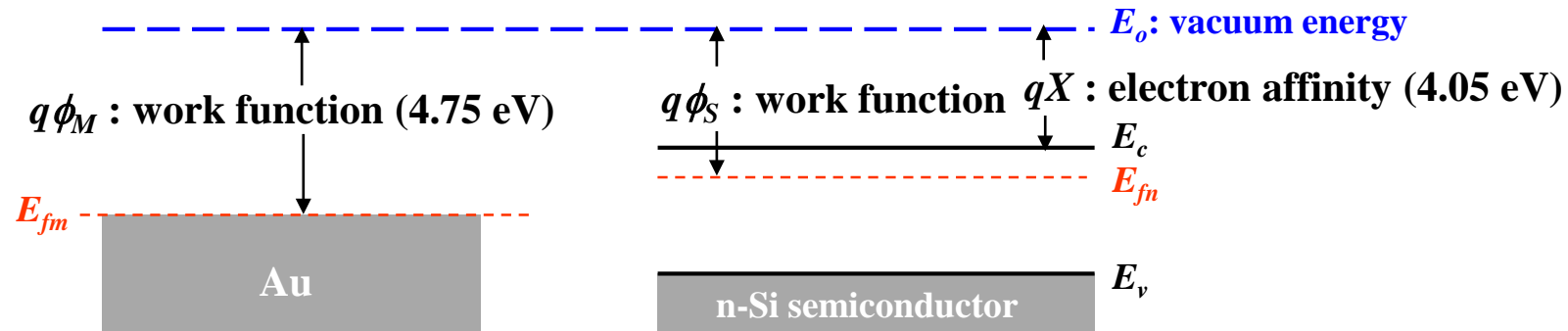
- **At thermal equilibrium, there is no net transfer of electrons at any energy. Therefore the Fermi level is constant throughout the system.**
- **If ϕ_M is larger than ϕ_S, e will flow from semiconductor to metal, and this will causes the E_c to bend upward.**
- **If ϕ_S is larger than ϕ_M, e will flow from metal to semiconductor, and this will causes the E_c to bend downward.**
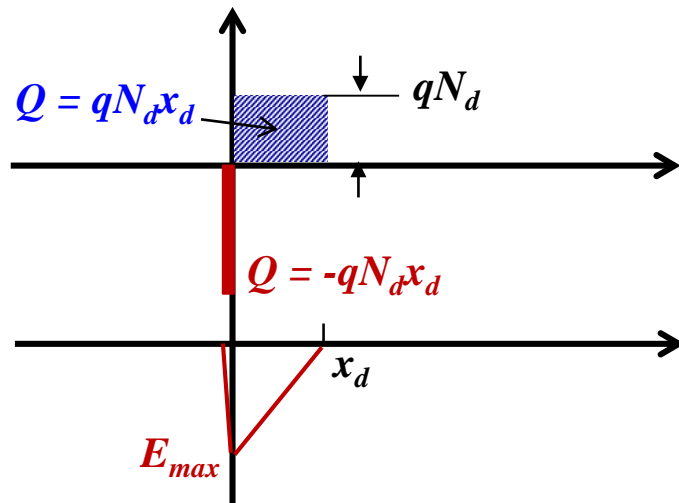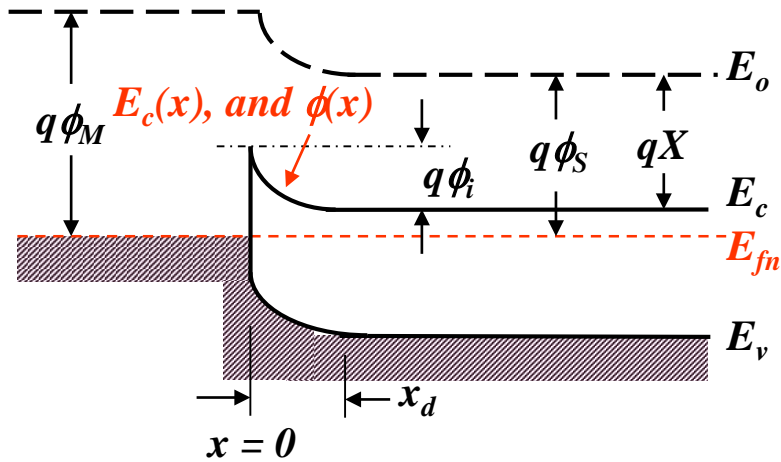
Band diagram of metal-semiconductor interface

n-semiconductor

p-semiconductor
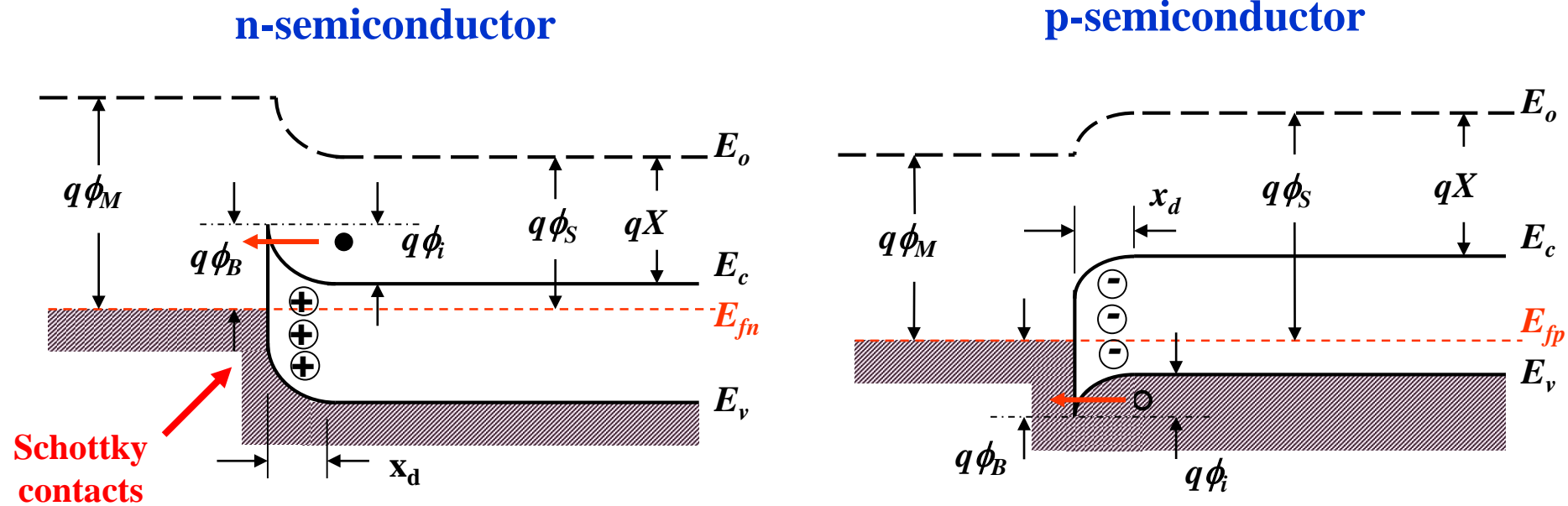
- ϕ_i : build-in protential = ϕ_M - ϕ_s = $\phi_M - X - (E_C - E_{fn})$
- ϕ_B : barrier height = $\phi_M - X$
- x_d : depletion region width
- Our development thus far has relied on the fact that basic band structure of the two materials are unchanged near the surface.

Charge, Depletion, and Capacitance

n-type semiconductor

Consider the Poisson equation:

$$\frac{d^2\phi}{dx^2} = -\frac{q}{\varepsilon}(N_d^+ - N_a^- + p - n)$$

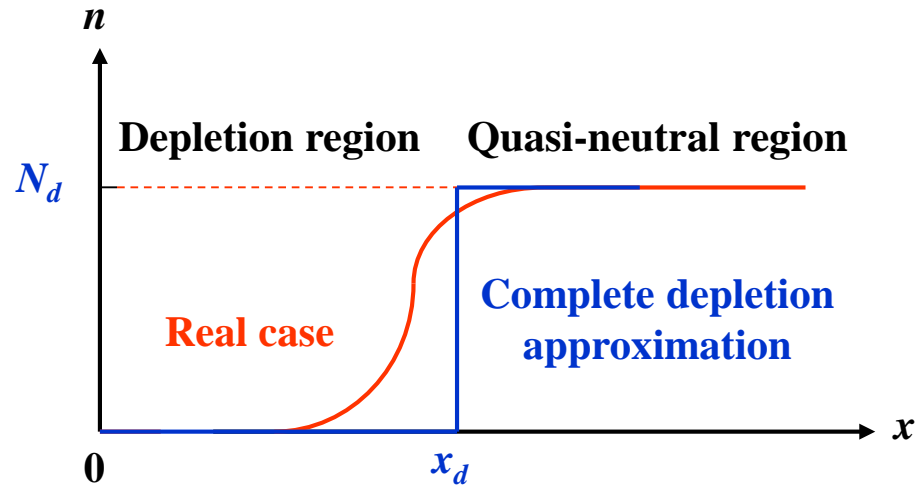**For $x < x_d$,**

$$n = N_c \times e^{-[E_C(x)-E_F]/kT}$$

For x >> 0,

$$n(\infty) = N_d = N_c \times e^{-(E_C(\infty)-E_F)/kT}$$

or $\quad n = N_d \times e^{-[E_C(x)-E_C(\infty)]/kT} = N_d \times e^{-q\phi(x)/kT}$

$$\frac{d^2\phi}{dx^2} = -\frac{q}{\varepsilon}N_d(1 - e^{-q\phi(x)/kT})$$

Complete Depletion Approximation

If ϕ_i is large enough to deplete the free carriers in the junction, the complete depletion **approximation** is then valid.
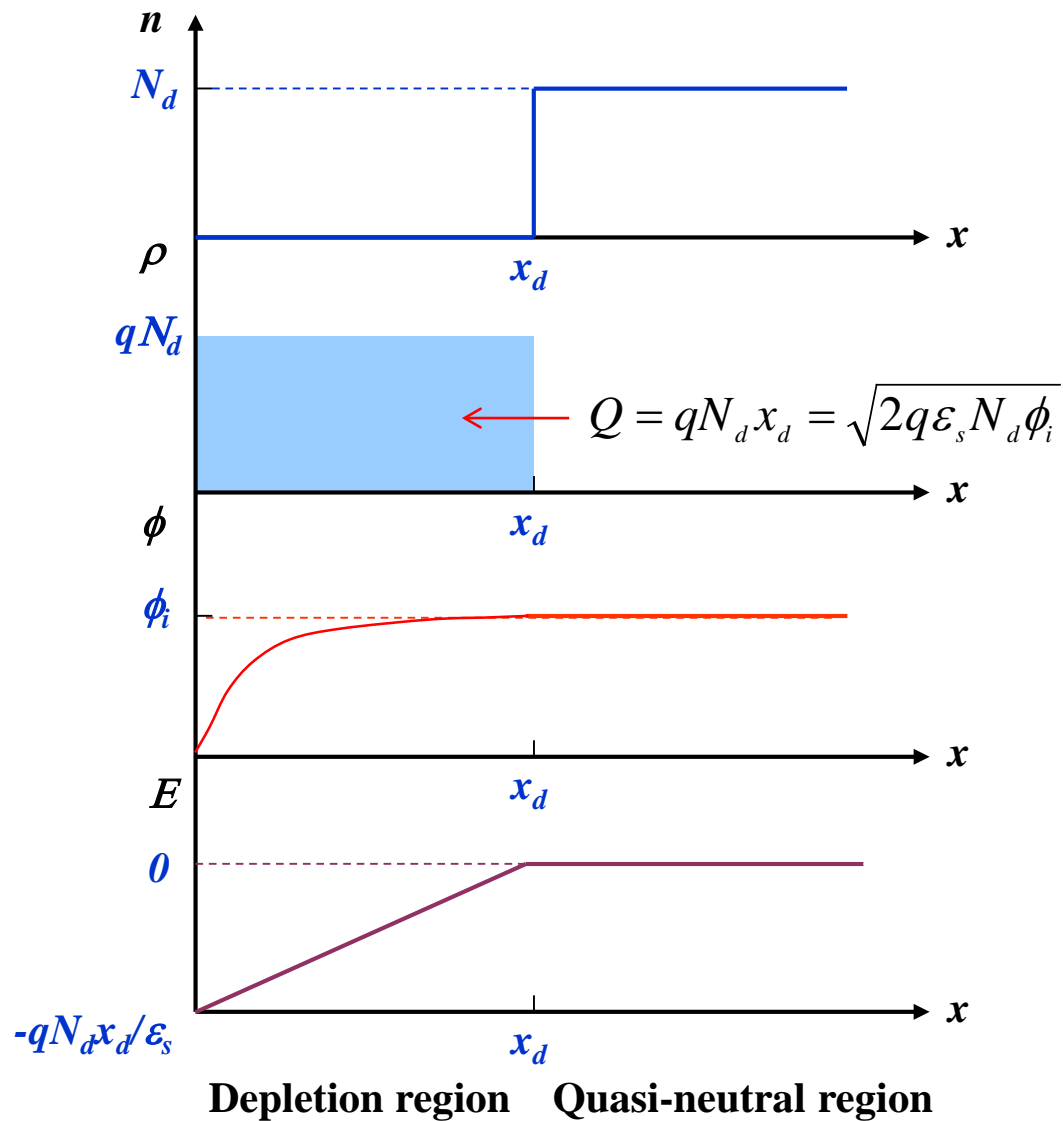
The Poisson equation becomes :

$$\frac{d^2\phi}{dx^2} = -\frac{q}{\varepsilon_s}N_d \quad \Rightarrow \quad \phi = -\frac{qN_d}{\varepsilon_s}\frac{x^2}{2} + C_1 x + C_2$$

Boundary conditions :

$$E(x_d) = -\frac{d\phi(x_d)}{dx} = 0 \Rightarrow C_1 = \frac{qN_d}{\varepsilon_s}x_d$$

$$\phi(0) = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow \quad \phi(x) = \frac{qN_d}{\varepsilon_s}\frac{x_d^2}{2}[1-(1-\frac{x}{x_d})^2]$$

$$\phi(x_d) = \phi_i = \frac{qN_d}{\varepsilon_s}\frac{x_d^2}{2} \quad or \quad x_d = \sqrt{\frac{2\phi_i\varepsilon_s}{qN_d}}$$

➡ **N_d** ↗ **x_d** ↘

$$E(x) = -\frac{d\phi(x)}{dx} = -\frac{qN_d}{\varepsilon_s}x_d(1-\frac{x}{x_d})$$

or $E_{max} = -\frac{qN_d}{\varepsilon_s}x_d$

Applied bias and capacitance

When a bias is applied to a metal-n-type semiconductor junction, we need to consider the non-equilibrium condition of the junction. Under a bias condition, the barrier is reduced when the metal is biased positively with respect to the semiconductor, and it is increased under bias of the opposite polarity.

$V_a = 0$

$V_a > 0$

$V_a < 0$
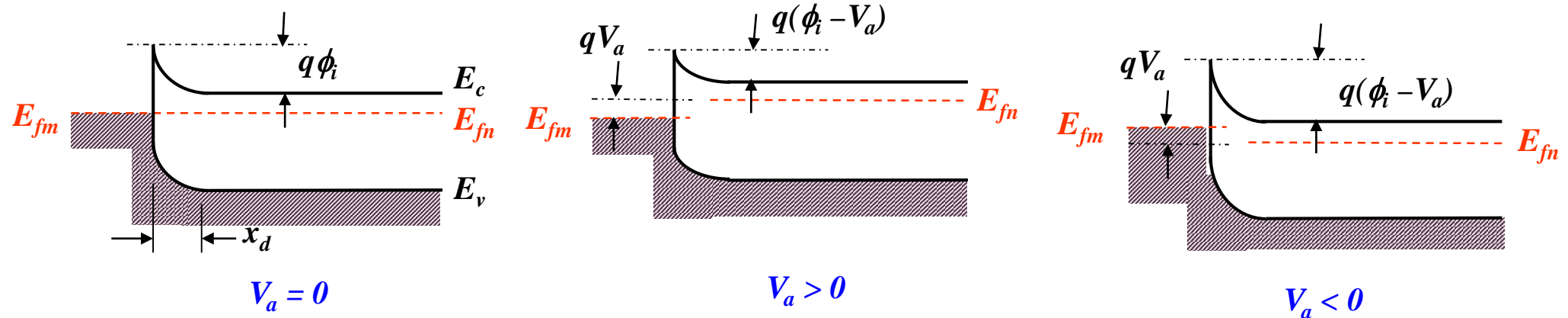
Under the bias conditions, the voltage drop across the space-charge region is changed to $(\phi_i - V_a)$, then the space charge density in the semiconductor changes accordingly.

$$Q = qN_d x_d = \sqrt{2q\varepsilon_s N_d (\phi_i - V_a)}$$

$$C = \frac{\partial Q}{\partial V_a} = \sqrt{\frac{q\varepsilon_s N_d}{2(\phi_i - V_a)}} = \frac{\varepsilon_s}{x_d} \quad or \quad (\phi_i - V_a) = \frac{q\varepsilon_s N_d}{2C^2}$$
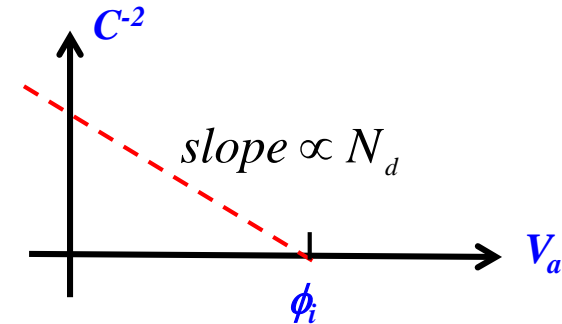
Example: Complete depletion approximation

Complete depletion approximation is valid only when the build-in potential is large enough to deplete the free carriers in the space-charge region.

In equilibrium,

$$J_n = 0 = q\mu_n n E_x + qD_n \frac{dn}{dx} \rightarrow E_x = -\frac{D_n}{\mu_n}\frac{1}{n}\frac{dn}{dx} = -\frac{kT}{q}\frac{1}{n}\frac{dn}{dx}$$

Use $E_x = -\frac{d\phi}{dx} \rightarrow d\phi = \frac{kT}{q}\frac{dn}{n} \rightarrow \Delta\phi = \frac{kT}{q}\ln\frac{n'}{n}$

For $n' = 10^{18}$ cm⁻³, $n = 10^{16}$ cm⁻³, and the variation of the dopant concentration is over the range of a few hundred nanometers,

$$\rightarrow \Delta\phi = 0.12V \rightarrow E_x = -\frac{\Delta\phi}{\Delta x} = 2400V/cm$$

$$\frac{d^2\phi}{dx^2} \approx 5\times10^7 V/cm^2 = -\frac{q}{\varepsilon_s}(N_D - n) \rightarrow |N_D - n| \approx 10^{15} cm^{-3} \rightarrow N_D \approx n >> 0$$

Under this condition, the complete depletion approximation is not valid.

I-V Characteristics

Drift-diffusion approach

$$J_x = q[n\mu_n E_x + D_n \frac{dn}{dx}] = qD_n[\frac{-qn}{kT}\frac{d\phi}{dx} + \frac{dn}{dx}]$$

$$\frac{D_n}{\mu_n} = \frac{kT}{q}$$

$$\times e^{-\frac{q\phi}{kT}}, \quad and \quad \int_0^{x_d}[....]dx \quad \Rightarrow J_x\int_0^{x_d} e^{-\frac{q\phi}{kT}}dx = qD_n[ne^{-\frac{q\phi}{kT}}]_0^{x_d}$$

B.C. : $\phi(x_d) = \phi_i - V_a = \phi_B - \phi_n - V_a$, and $\phi(0) = 0$,

$$n(x_d) = N_d = N_C \cdot e^{-q\phi_n/kT}, \quad and \quad n(0) = N_C \cdot e^{-q\phi_B/kT}$$

$$\Rightarrow J_x = \frac{qD_n N_C e^{-\frac{q\phi_B}{kT}}}{\int_0^{x_d} e^{-\frac{q\phi(x)}{kT}} dx} \times [e^{\frac{qV_a}{kT}} - 1] \quad \text{where} \quad \phi(x) = \frac{qN_d}{\varepsilon_s}\frac{x_d^2}{2}[1 - (1 - \frac{x}{x_d})^2]$$
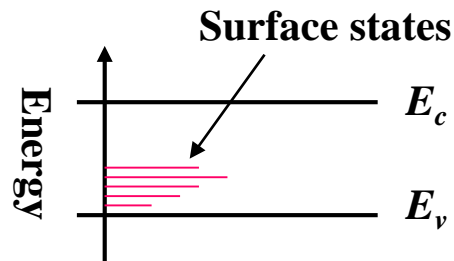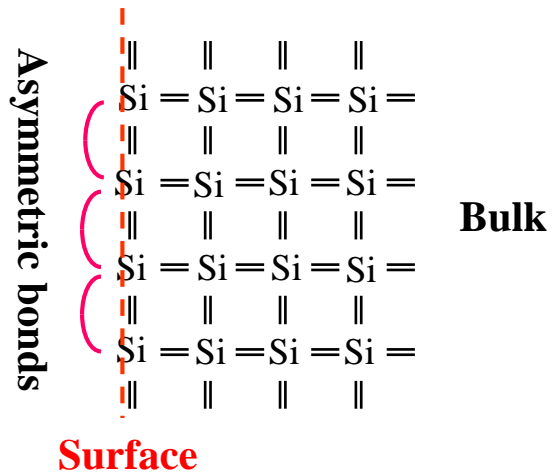
$$J_x = J_s \times [e^{\frac{qV_a}{kT}} - 1]$$

where $J_s = \dfrac{q D_n N_C e^{-\frac{q\phi_B}{kT}}}{kT} \times [\dfrac{2q(\phi_i - V_a)N_d}{\varepsilon_s}]^{1/2}$

For convenience, $J_x = J_s' \times [e^{\frac{qV_a}{nkT}} - 1]$ **1.02 < *n* < 1.15, n: ideality factor**

Surface States

There are extra allowed states for electrons that are present at the semiconductor surface, but not within the bulk.
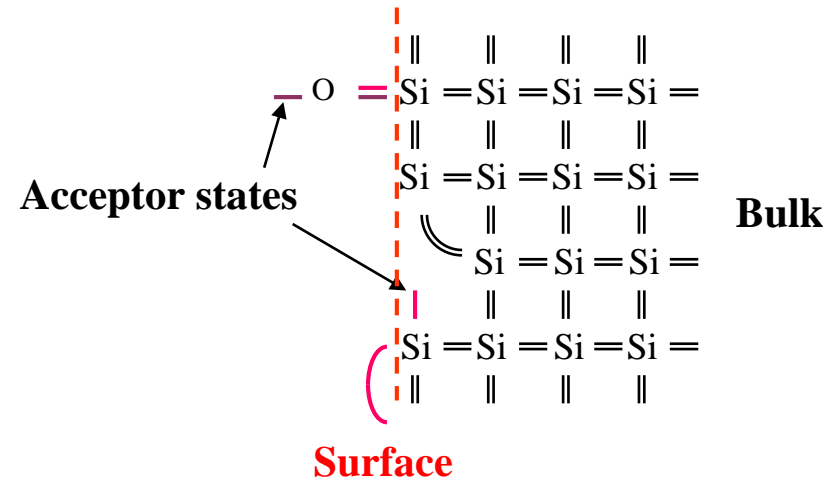
Tamm or Shockley states

Bonded foreign atoms or crystal defects

$N_{st} \sim N_o^{2/3}$ **(cm⁻²), for Si, $N_o = 5 \times 10^{22}$ cm⁻³**

$\longrightarrow N_{st} = 10^{15}$ **cm⁻²**

● **Donor states: states are neutral when occupied by electrons and positively charged when unoccupied.**

● **Acceptor states: states are negative when occupied and neutral when empty.**

Example: Surface state and surface recombination velocity

An n-type Si wafer with a doping concentration of 10^{18} cm^{-3} at RT is illuminated through a passivating layer of SiO$_2$ so that the hole and electron density exceed their thermal equilibrium values. Assume that the surface electron density $n_s = 10^{16}$ cm^{-3} at thermal equilibrium, the photo generation of carriers is 2×10^{14} cm^{-2}s^{-1}, and the surface hole density under illumination increases to $p_s = 10^{10}$ cm^{-3}.
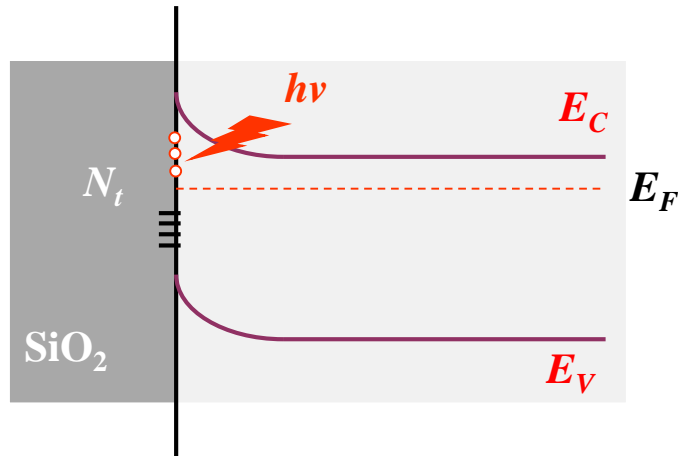
Since $n_{so} = 10^{16}$ cm^{-3} << $N_d = 10^{18}$ cm^{-3}, the surface is slightly depleted of majority carriers.

The thermal equilibrium density of hole is

$$p_{so} = n_i^2/n_{so} = 2.1\times10^4$$

After photo generation,

$$p_s = 10^{10} \text{ cm}^{-3} >> p_{so},$$

therefore the excess carrier density of hole is $\Delta p_s = 10^{10}$ cm^{-3}

If $S_p = 10^4$ cm/s, the net recombination rate is

$$U_s = S_p \times \Delta p_s = 10^{14} \text{ cm}^{-2}\text{s}^{-1},$$

therefore 50% of the excess carriers are recombined at the surface.

Part II – Optics

Maxwell's equations and wave propagation

Maxwell's equations, in MKS unit, can be written in the form

$$\nabla \times \vec{E}(\vec{r},\omega) = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H}(\vec{r},\omega) = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \nabla \cdot \vec{B} = 0 \qquad and \quad \nabla \cdot \vec{H} = \rho$$

$$\vec{E} : V/m \qquad \vec{H} : A/m \qquad \vec{B} : \frac{V \cdot \sec}{m^2} = \frac{Web}{m^2} \qquad \vec{D} : \frac{C}{m^2} \qquad \rho : \frac{C}{m^3} \quad and \qquad \vec{J} : \frac{A}{m^2}$$

Continuity equation: $\qquad \nabla \cdot \vec{J}(\vec{r},\omega) = -\frac{\partial \rho}{\partial t}$

Totally, we have 15 unknowns, but with only 6 independent equations. To solve the Maxwell's equations, we need more equations.

➡️ **Constitutive relation**

$$\vec{J}(\vec{r},\omega) = f\{\vec{E}(\vec{r},\omega), \vec{B}(\vec{r},\omega)\} \qquad \vec{D}(\vec{r},\omega) = f\{\vec{E}(\vec{r},\omega), \vec{B}(\vec{r},\omega)\} \qquad and$$

$$\vec{H}(\vec{r},\omega) = f\{\vec{E}(\vec{r},\omega), \vec{B}(\vec{r},\omega)\}$$

In a linear lossless, homogeneous, and isotropic medium, the constitutive relation can be written as

$$\vec{J}(\vec{r},\omega) = 0 \qquad \vec{D}(\vec{r},\omega) = \varepsilon(\omega)\vec{E}(\vec{r},\omega) \qquad and \qquad \vec{H}(\vec{r},\omega) = \frac{\vec{B}(\vec{r},\omega)}{\mu(\omega)}$$

Additional 6 equations

Monochromatic plane waves

Consider a homogenous and source free region in free space, then

$$\vec{J}(\vec{r},\omega) = 0 \qquad \vec{D}(\vec{r},\omega) = \varepsilon_0 \vec{E}(\vec{r},\omega) \qquad \vec{B}(\vec{r},\omega) = \mu_0 \vec{H}(\vec{r},\omega)$$

For a monochromatic plane wave,

$$\vec{E}(\vec{r},\omega) = \mathrm{Re}[\vec{E}(\vec{r})e^{-i\omega t}] \qquad and \qquad \vec{H}(\vec{r},\omega) = \mathrm{Re}[\vec{H}(\vec{r})e^{-i\omega t}]$$

$$\nabla \times \vec{E} = i\omega\mu_0 \vec{H} \qquad \nabla \times \vec{H} = -i\omega\varepsilon_0 \vec{E} \qquad and \quad \nabla \cdot \vec{E} = \nabla \cdot \vec{H} = 0$$

Consider now,

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = i\omega\mu_0 \nabla \times \vec{H} = \omega^2 \mu_0 \varepsilon_0 \vec{H}$$

0

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \qquad \nabla^2 \vec{H} + k^2 \vec{H} = 0 \qquad and \qquad k^2 = \omega^2 \mu_0 \varepsilon_0 = \omega^2 c^2$$

Wave equations

The solutions to above wave equations can be written as

$$\vec{E} = \vec{E}_f e^{i\vec{k}\cdot\vec{r}} + \vec{E}_b e^{-i\vec{k}\cdot\vec{r}} \qquad and \qquad \vec{H} = \vec{H}_f e^{i\vec{k}\cdot\vec{r}} + \vec{H}_b e^{-i\vec{k}\cdot\vec{r}} \qquad \textbf{or}$$

$$\vec{E}(\vec{r},t) = \vec{E}_f \cos(\omega t - \vec{k}\cdot\vec{r}) + \vec{E}_b \cos(\omega t + \vec{k}\cdot\vec{r})$$

Phase functions

If we only consider the forward-propagating waves,

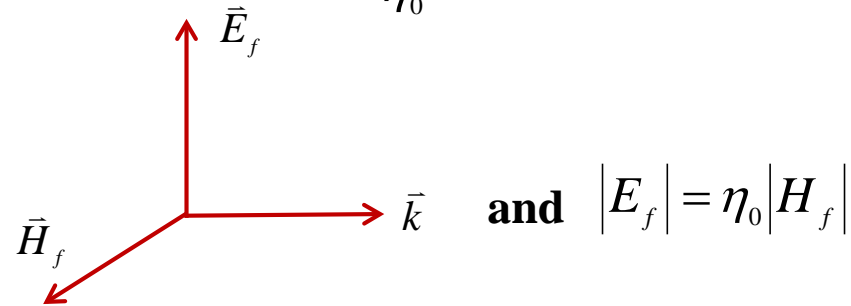$$\vec{E} = \vec{E}_f e^{i\vec{k}\cdot\vec{r}} \qquad and \qquad \vec{H} = \vec{H}_f e^{i\vec{k}\cdot\vec{r}}$$
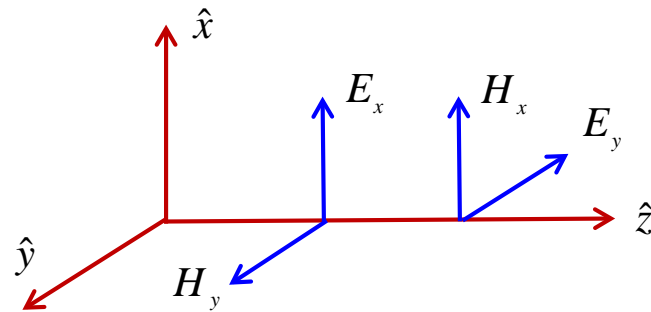
Use $\nabla \times \vec{E} = i\omega\mu_0\vec{H}$

$$i\vec{k} \times \vec{E}_f e^{i\vec{k}\cdot\vec{r}} = i\omega\mu_0\vec{H}_f e^{i\vec{k}\cdot\vec{r}} \rightarrow \frac{\vec{k}}{k} \times \vec{E}_f = \sqrt{\frac{\mu_0}{\varepsilon_0}}\vec{H}_f \rightarrow \hat{e}_k \times \vec{E}_f = \sqrt{\frac{\mu_0}{\varepsilon_0}}\vec{H}_f = \eta_0\vec{H}_f$$

377Ω

Similarly, $\quad \hat{e}_k \times \vec{H}_f = -\frac{1}{\eta_0}\vec{E}_f \quad$ *and* $\quad \hat{e}_k \cdot \vec{H}_f = \hat{e}_k \cdot \vec{E}_f = 0$

and $\quad |E_f| = \eta_0|H_f|$

Consider now, if the wave is propagating along z direction

$$|E_x| = \eta_0|H_y| : x-polarized \quad and \quad |E_y| = -\eta_0|H_x| : y-polarized$$
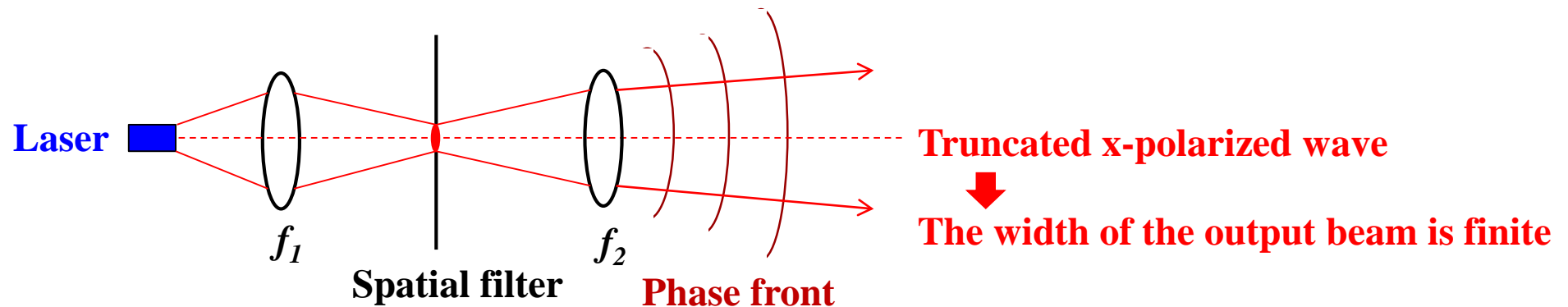
Example: Plane waves

What is a plane wave?

The wave has a perfect flat and infinite in extent phase front. ← **Does not exit!**

So under what condition, an obtainable wave can be approximated as a plane wave?

Laser f_1 **Spatial filter** f_2 **Phase front**

Truncated x-polarized wave

⬇

The width of the output beam is finite

Let the truncated x-polarized wave be $\vec{E} = \hat{e}_x E_x e^{ik\cdot z} e^{-\frac{r^2}{2w_o^2}}$ $w_o : beam\_waist$

For such a plane wave, $E_y = H_x = E_z = H_z = 0$, and $E_x = \eta H_y$

However, we always have $\nabla \cdot \vec{E} = 0 = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z}$

$\qquad\qquad\qquad\qquad\qquad\qquad$ **?** $\qquad$ **0** $\qquad$ **0**

If the wave is truncated, $\dfrac{\partial E_x}{\partial x} \neq 0$

➡ $\quad \vec{E} = \hat{e}_x E_x + \hat{e}_z E_z \qquad and \quad \vec{H} = \hat{e}_y H_y + \hat{e}_z H_z$

Consider the Poynting vector, $\vec{S} = \vec{E} \times \vec{H} = \hat{e}_z E_x H_y - \hat{e}_y E_x H_z - \hat{e}_x E_z H_y$

➡ **There must be radial components of the Poynting vector. This effect is also know as diffraction spreading.**
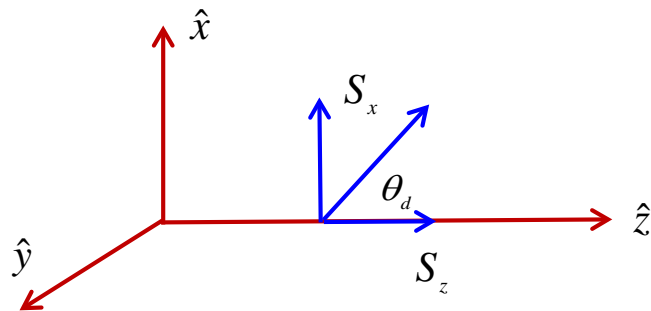
Polarization

Assuming a plane wave propagating along z direction,

$$\vec{E}_t = \hat{e}_x E_x(\vec{r},t) + \hat{e}_y E_y(\vec{r},t)$$

$$and \quad E_x(\vec{r},t) = a_x \cos(\omega t - kz + \varphi_x) \qquad E_y(\vec{r},t) = a_y \cos(\omega t - kz + \varphi_y)$$

Let $\tau = \omega t - kz$, then

$$\frac{E_x}{a_x} = \cos\tau \cdot \cos\varphi_x - \sin\tau \cdot \sin\varphi_x \quad and \quad \frac{E_y}{a_y} = \cos\tau \cdot \cos\varphi_y - \sin\tau \cdot \sin\varphi_y$$
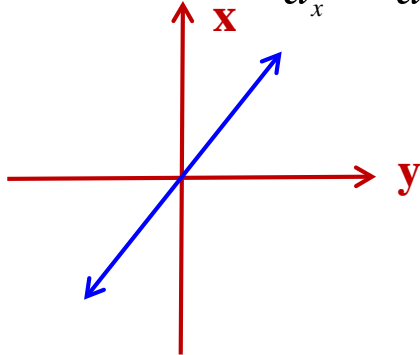
Eliminating τ from above equation,

$$\Rightarrow \quad (\frac{E_x}{a_x})^2 + (\frac{E_y}{a_y})^2 - 2\frac{E_x E_y}{a_x a_y}\cos\varphi = \sin^2\varphi \quad and \quad \varphi = \varphi_x - \varphi_y$$

Example: Polarization

$$\left(\frac{E_x}{a_x}\right)^2 + \left(\frac{E_y}{a_y}\right)^2 - 2\frac{E_x E_y}{a_x a_y}\cos\varphi = \sin^2\varphi \quad and \quad \varphi = \varphi_x - \varphi_y$$
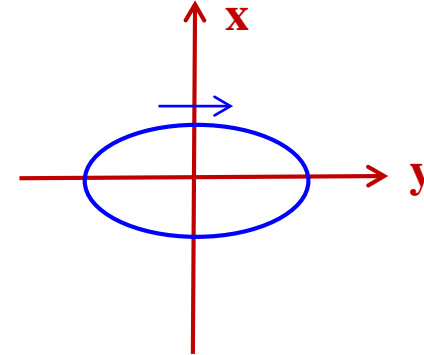
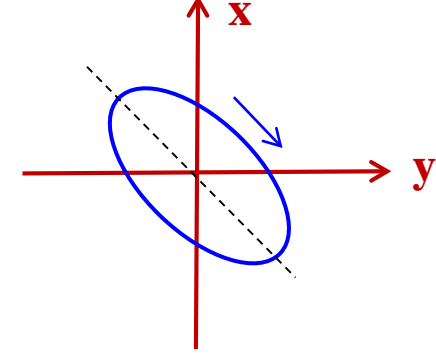**I.** $\varphi = 0$ ➡ $\dfrac{E_x}{a_x} = \dfrac{E_y}{a_y}$
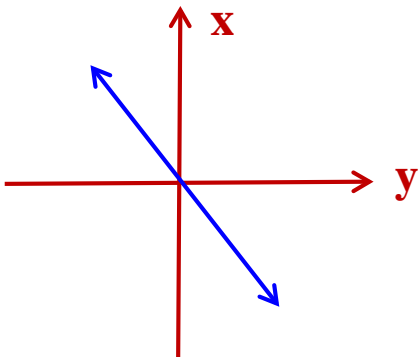
II. $\varphi = \pi/4$
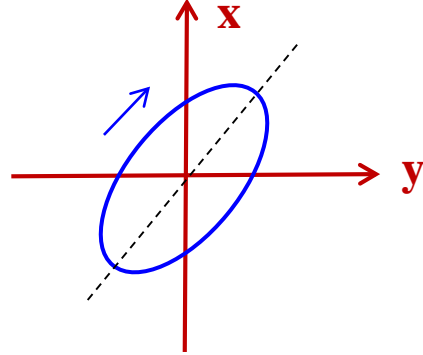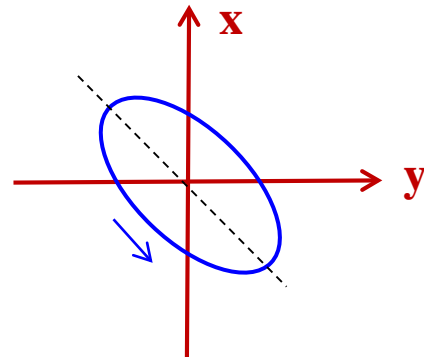
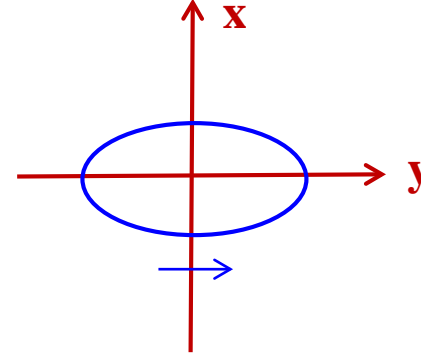**III.** $\varphi = \pi/2$
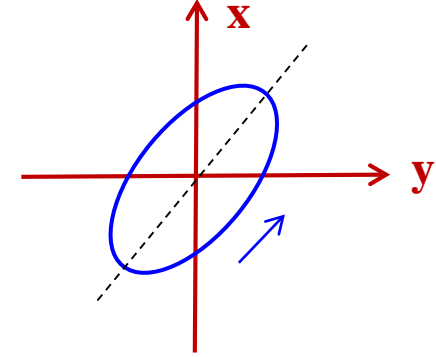
IV. $\varphi = 3\pi/4$
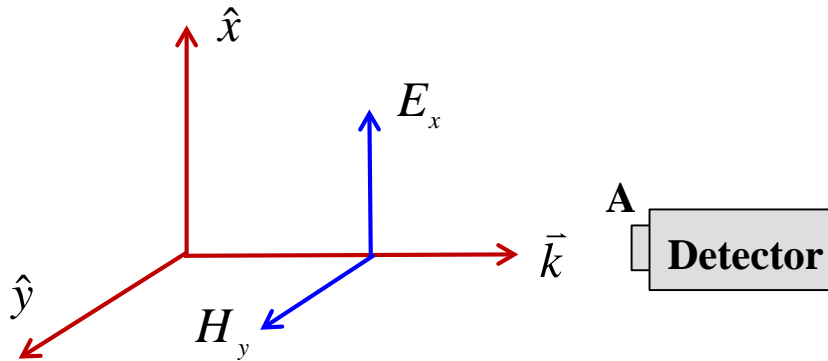
V. $\varphi = \pi$

VI. $\varphi = 5\pi/4$

VII. $\varphi = 3\pi/2$

VIII. $\varphi = \pi/4$

Detection
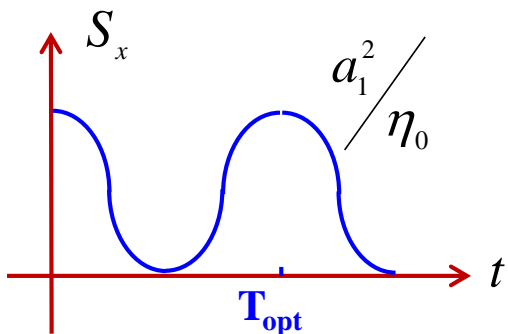

$$E_x = a_1 \cos(\omega t - kz) \quad and \quad E_y = a_2 \cos(\omega t - kz + \varphi)$$

$$H_y = \frac{a_1}{\eta_0} \sin(\omega t - kz) \quad and \quad H_x = -\frac{a_2}{\eta_0} \sin(\omega t - kz + \varphi)$$

$$\vec{S} = \vec{E} \times \vec{H} = \hat{e}_z \{\frac{a_1^2}{2\eta_0}[1 + \cos(2\omega t - 2kz)] + \frac{a_2^2}{2\eta_0}[1 + \cos(2\omega t - 2kz + 2\varphi)]\}$$

So energy flows of the two polarization states simply add up together and no cross terms.

Since no detector can react fast enough at optical frequency, so what we actually see is the average energy across A.

$$\vec{S} = \vec{E} \times \vec{H} = \hat{e}_z \{ \frac{a_1^2}{2\eta_0}[1 + \cos(2\omega t - 2kz)] + \frac{a_2^2}{2\eta_0}[1 + \cos(2\omega t - 2kz + 2\varphi)] \}$$

Take < >,

$$< \vec{S}(z,t) > = \frac{1}{2\tau_d} \int_{t-\tau_d}^{t+\tau_d} \vec{S}(z,t)dt \qquad \tau_d : \det ector\_response\_time$$

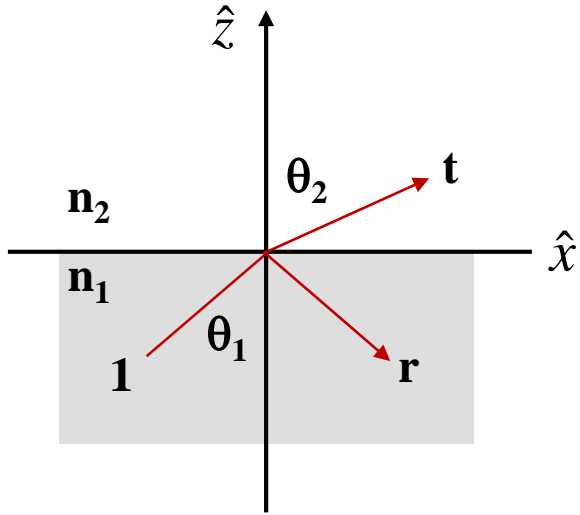Since τ_d >> T_{opt}, so the response time average ~ infinite time average.

$$< \vec{S}(z,t) > = \lim \frac{1}{2T} \int_{-T}^{T} \vec{S}(z,t)dt = (\frac{a_1^2}{2\eta_0} + \frac{a_2^2}{2\eta_0})\hat{e}_z$$

Therefore the energy flow is directly proportional to the time average of $\vec{E}(t) \cdot \vec{E}(t)$.

$$S_{ave} = < S(t) > = \frac{1}{2}\operatorname{Re}(\vec{E} \times \vec{H}^*) = \frac{1}{2}EH^* = \frac{E^2}{2\eta_0}$$

Example: Transmission and reflection coefficients

$$t_\perp = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \quad and \quad r_\perp = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \quad \rightarrow TE$$
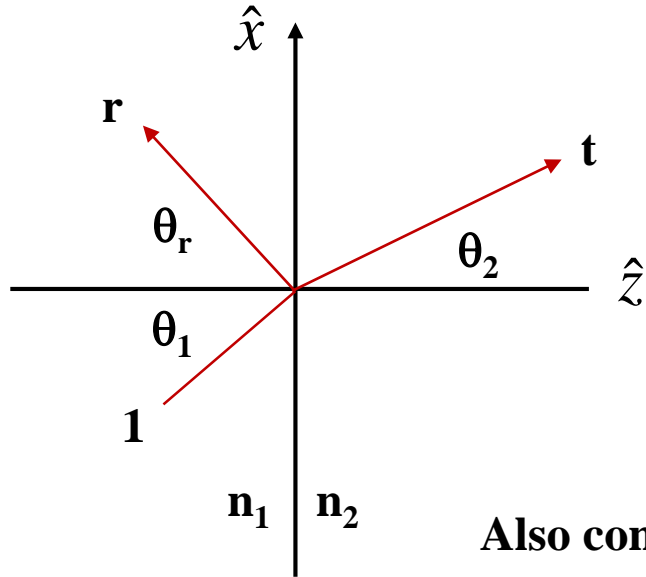
$$t_{//} = \frac{2 n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \quad and \quad r_{//} = \frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \quad \rightarrow TM$$

$$\vec{E}_r = r_{\perp,//} \vec{E}_i \qquad \vec{E}_t = t_{\perp,//} \vec{E}_i \quad and \quad R_{\perp,//} = \left| r_{\perp,//} \right|^2 \qquad T_{\perp,//} = \left| t_{\perp,//} \right|^2$$

Note that $\quad R + T \xrightarrow{TE} \left| r_\perp \right|^2 + \left| t_\perp \right|^2 = \dfrac{5 n_1 - 2 n_1 n_2 + n_2^2}{(n_1 + n_2)^2} = 1_{only\_if\_n1=n2}$

Brewster angle

Snell law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, if $n_2 > n_1 \rightarrow \sin\theta_1 > \sin\theta_2$

➡️ **$\cos\theta_1 < \cos\theta_2$ and $n_1\cos\theta_1 < n_2\cos\theta_2$**

From $\quad t_{//} = \dfrac{2n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \quad and \quad r_{//} = \dfrac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$
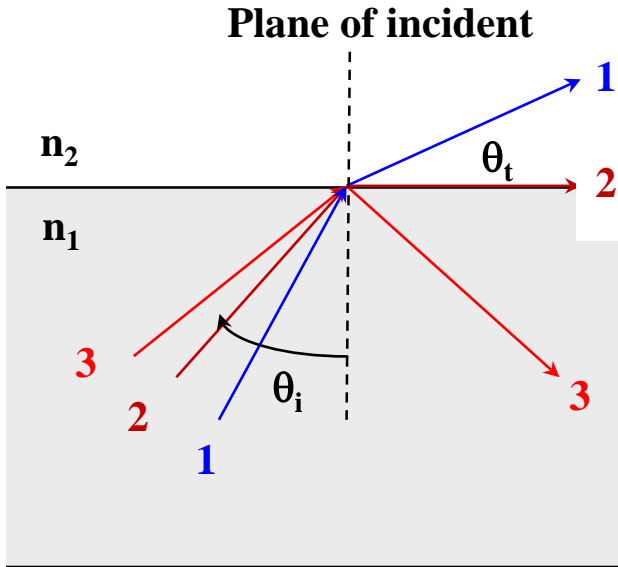
Also consider that $r_{//} = 0$ if $n_2\cos\theta_1 = n_1\cos\theta_2 \rightarrow$ $\dfrac{\cos\theta_2}{\cos\theta_1} = \dfrac{n_2}{n_1}$

Therefore there must exist the condition, when $\theta_1 = \theta_B$

$\rightarrow \theta_1 = \theta_r \rightarrow \theta_r + \theta_2 = \pi/2 \rightarrow \tan\theta_B = n_2/n_1$

So when $\theta_1 = \theta_B \rightarrow r_{//} = 0$, indicating no reflection and total transmission.

Total internal reflection (TIR)

Plane of incident

Snell's law: $\quad n_1 \sin\theta_i = n_2 \sin\theta_t \quad n_1 > n_2$

Increasing θ_i will also increase θ_t. When $\theta_i = \theta_{i2} = \theta_c \rightarrow \theta_{t2} = 90^o$, or
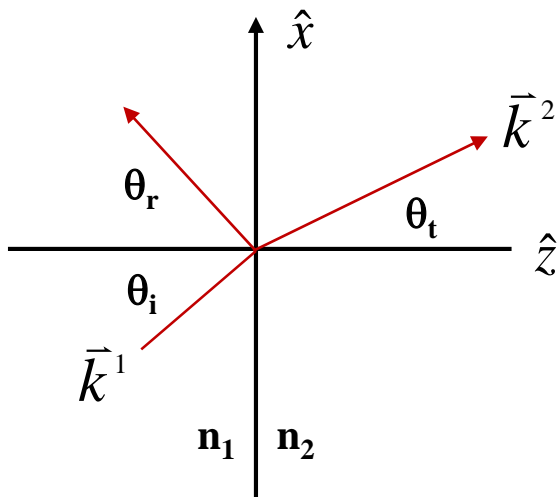
$$n_1 \sin\theta_{i2} = n_2 \quad \blacktriangleright \quad \theta_{i2} \equiv \theta_c = \sin^{-1}\frac{n_2}{n_1}$$

If $\theta_i > \theta_c$, $\quad n_1 \sin\theta_i = n_2 \sin\theta_t \quad or \quad \sin\theta_t = \frac{n_1}{n_2}\sin\theta_i > 1$

$$\vec{k}_{1x} = \vec{k}_{2x} \qquad Snell'\_law \qquad \textbf{and} \qquad \vec{k}_2^2 = \vec{k}_{2x}^2 + \vec{k}_{2z}^2 = \vec{k}_{1x}^2 + \vec{k}_{2z}^2 = \vec{k}_1^2 \sin^2\theta_i + \vec{k}_{2z}^2$$

$\blacktriangleright \quad \vec{k}_{2z}^2 = \vec{k}_2^2 - \vec{k}_1^2 \sin^2\theta_i = \omega^2\mu\varepsilon_2 - \omega^2\mu\varepsilon_1 \sin^2\theta_i = \omega^2\mu\varepsilon_2[1-(\frac{n_1}{n_2}\sin\theta_i)^2]$

$\blacktriangleright \quad k_{2z} = -i\omega\sqrt{\mu\varepsilon_2}\sqrt{(\frac{n_1}{n_2}\sin\theta_i)^2 - 1} = -ik_2\sqrt{(\frac{n_1}{n_2}\sin\theta_i)^2 - 1}$

Example: Evanescence waves in z direction

$$\vec{E}_t^{TE} = t \times \vec{E}_i e^{i\omega t} e^{-i\vec{k}\cdot\vec{r}} = t \times \vec{E}_i e^{i\omega t} e^{-ik_{2x}\cdot x} e^{-ik_{2z}\cdot z}$$ **Since** $\quad k_{2z} = -ik_2\sqrt{(\dfrac{n_1}{n_2}\sin\theta_i)^2 - 1}$

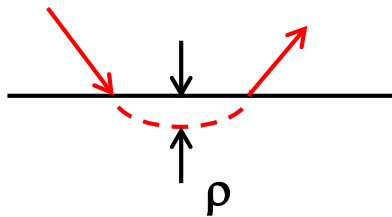$$= t \times E_i e^{i\omega t} e^{ik_{2x}x\sin\theta_t} e^{-k_2\sqrt{(\sin^2\theta_t - 1)}z}$$

$$= t \times E_i e^{i\omega t} \underbrace{e^{ik_{2x}x\sin\theta_t}}_{\text{Propagating wave}} \underbrace{e^{-k_2\sqrt{(\sin^2\theta_t - 1)}z}}_{\text{Evanescent wave}}$$ *or* $\boxed{\rho = \dfrac{\lambda_2}{2\pi}\dfrac{1}{\sqrt{(\sin^2\theta_t - 1)}}}$
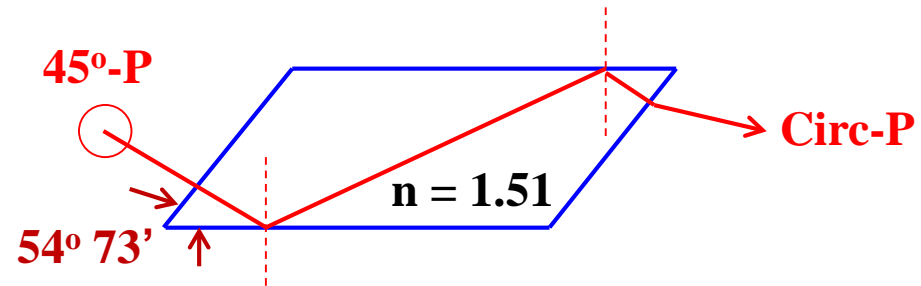
Goos-Hanschen depth

So we have a propagating wave along x direction (real $k_{1x} = k_{2x}$) and an evanescent wave along z direction (imaginary k_{2z}).
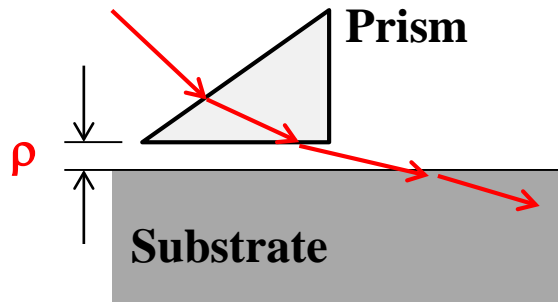
Example: TIR applications

I. Circular polarization

45º-P

Circ-P

n = 1.51

54º 73'

II. Prism coupler

Prism

ρ

Substrate

III. Optical waveguide

TIR

n_2

n_1

n_2

$\theta_i > \theta_c$

$\theta_i < \theta_c$